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RESEARCH ARTICLE

Essential equilibria of large generalized games

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Abstract We characterize the essential stability of games with a continuum of players, where strategy profiles may affect objective functions and admissible strategies. Taking into account the perturbations defined by a continuous mapping from a complete metric space of parameters to the space of continuous games, we prove that essential stability is a generic property and every game has a stable subset of equilibria. These results are extended to discontinuous large generalized games assuming that only payoff functions are subject to perturbations. We apply our results in an electoral game with a continuum of Cournot-Nash equilibria, where the unique essential equilibrium is that only politically engaged players participate in the electoral process. In addition, employing our results for discontinuous games, we determine the stability properties of competitive prices in large economies.

Keywords Large generalized games · Essential equilibria · Essential sets and components

JEL Classification C62 · C72 · C02

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1 Introduction

In this study, we focus on the essential stability of Cournot-Nash equilibria for large generalized games, analyzing how equilibrium strategies change when some characteristics of the game are perturbed. We allow any kind of perturbation, provided that it can be defined through a continuous parametrization of the space of games over a complete metric space of parameters.

We consider large generalized games, i.e., games with a continuum of non-atomic players and a finite number of atomic players, where strategy profiles may affect players' objective functions and admissible strategies. Decisions made by non-atomic players are codified and aggregated, generating messages to the participants to the game. Under mild conditions on the characteristics of the game, a pure strategy Cournot-Nash equilibrium always exists [see Balder (1999, 2002); Carmona and Podczeck (2014)].

In this context, it is natural to ask how equilibrium strategies of atomic players and equilibrium messages induced by decisions made by non-atomic players—the pieces of information that fully determine the players' strategic behavior—change when the characteristics of the game are perturbed. We focus on essential stability, that is, we determine conditions under which the Cournot-Nash equilibria of a game can be approximated by equilibria of perturbed games.

We begin our analysis of essential stability assuming that games are continuous[1] and any of their characteristics can be perturbed, i.e., objective functions, action sets, or correspondences of admissible strategies. In this context, there is a dense residual subset of the space of large generalized games in which messages and atomic players' strategies associated with Cournot-Nash equilibria are stable to perturbations (Theorem 1).[2] In particular, uniqueness of equilibrium messages and strategies for atomic players is a sufficient condition for stability. We also analyze the stability of subsets of equilibrium messages and actions, obtaining analogous results to those ensured in the literature for convex games with finitely many players: every game has essential subsets of Cournot-Nash equilibria (Theorem 2).

These stability results are extended to allow a broader range of perturbations, which we capture through the parametrizations of the space of games. If the set of parameters that can be perturbed constitutes a complete metric space and the mapping associating these parameters with large generalized games is continuous, then stability results previously described still hold (Theorem 3) and essential sets are stable too (Theorem 4).

If we do not assume continuity of objective functions, and admissible strategies are subject to perturbations, then the space of large generalized games with a non-empty set of equilibria is not necessarily complete, which is a crucial property to guarantee our previous results. For this reason, we extend the analysis to discontinuous large generalized games assuming that only payoff functions can be perturbed. To ensure equilibrium existence, we follow the results of Carmona and Podczeck (2014), which extended the model of Balder (2002) to the discontinuous case. We focus on large generalized games where players have upper hemicontinuous correspondences

[1] That is, for every player, objective functions and correspondences of admissible strategies are continuous.

[2] A subset of a metric space is residual if it contains the intersection of a countable family of dense and open sets.

of admissible strategies and both payoff functions of non-atomic players and the sum of atomic players' payoff functions are upper semicontinuous. In this context, we prove that the collection of generalized payoff secure games [see Carbonell-Nicolau (2010); Carmona and Podczeck (2014, Definition 4)] is a complete metric space. Hence, when perturbations on payoff functions can be captured through continuous parametrizations, Cournot-Nash equilibria are generically essential and any large game has essential subsets of equilibria that are stable (Theorem 5). Since our model includes finite-player games as a particular case, our findings about stability for discontinuous games complement the previous results of Yu (1999) and Carbonell-Nicolau (2010).

To obtain our results about essential stability, we prove that the compact-valued correspondence associating each generalized game with its set of equilibrium messages–actions, called Cournot-Nash correspondence, has a closed graph. To guarantee this property, we use the fact that the set of non-atomic players has finite measure and their strategies are transformed into finite-dimensional messages. Indeed, under these conditions, we ensure the closed graph property of the Cournot-Nash correspondence by applying the multidimensional Fatou's Lemma [see Hildenbrand (1974, Lemma 3, page 69)].

We illustrate our results through some applications. Since essential stability can be viewed as a refinement of equilibrium, we consider an electoral game in which the unique essential equilibrium is the one consistent with the participation of only politically engaged individuals. Also, based on our results for discontinuous games, we analyze the stability of equilibrium prices in large economies.

The rest of the paper is organized as follows. Section 2 discusses the related literature. In Sect. 3, we describe the space of large generalized games. In Sects. 4 and 5, we analyze essential stability properties of Cournot-Nash equilibria when objective functions and correspondences of admissible strategies are continuous. In Sect. 6, we apply our results to an electoral game. In Sect. 7, we extend our model to include some classes of discontinuous games, and we apply these extensions in Sect. 8 to analyze the stability of prices in large competitive markets. The proofs of our main results are given in the Appendix.

2 Related literature

The concept of essential stability has its origins in mathematical analysis literature, where it was introduced as a property of fixed points of functions and correspondences. In a seminal paper, Fort (1950) introduces the concept of essential fixed point of a function: a fixed point is essential if it can be approximated by fixed points of functions close to the original one. In addition, a function is essential if it has only essential fixed points. Considering the set of continuous functions from a compact metric space to itself, Fort (1950) proves that the set of essential functions is dense. He also proves that any continuous function which has only one fixed point is essential. These concepts and properties have natural extensions to multivalued mappings, as shown by Jiang (1962). Since not all mappings are essential, it is natural to analyze the stability of subsets of fixed points. With this aim, Kinoshita (1952) introduces the concept of essential component of the set of fixed points of a function: a maximal connected set that is stable to perturbations on the characteristics of the function. He proves that any continuous mapping has at least one essential component of fixed points. Jiang (1963) and Yu and Yang (2004) extend these results to multivalued mappings, by proving that compact-valued upper hemicontinuous correspondences have at least one essential component. These results are complemented by Yu et al. (2005) who also analyze how essential components change when mappings are perturbed.

This literature motivates the study of equilibrium stability in games. As in every noncooperative game the set of Nash equilibria coincides with the set of fixed points of the aggregate best response correspondence, the techniques described above allow to analyze how the equilibria of a game change when payoffs and action sets are perturbed. In this direction, essential stability of Nash equilibria of games with finitely many players is studied by Wu and Jiang (1962), Yu (1999), Yu et al. (2005), Zhou et al. (2007), Yu (2009), Carbonell-Nicolau (2010), and Scalzo (2013).

More precisely, Wu and Jiang (1962) address the stability of the set of Nash equilibria for games with a finite number of players and pure strategies. They ensure that any game can be approximated by a game whose equilibria are all essential. Yu (1999) formalizes and extends these results for convex games with a finite number of players and infinitely many strategies, analyzing perturbations in payoffs, action sets, and correspondences of admissible strategies. Jiang (1963), Yu et al. (2005), and Yu (2009) analyze the existence of essential components of the set of Nash equilibria for games and generalized games. Zhou et al. (2007) study the notion of essential stability for mixed-strategy equilibria in games with continuous payoff functions, compact sets of pure strategies, and finitely many players. They also compare the concept of essential stability with strategic stability, a notion studied by Kohlberg and Mertens (1986), Hillas (1990), and Al-Najjar (1995). Allowing for discontinuities on objective functions, Yu (1999), Carbonell-Nicolau (2010), and Scalzo (2013) analyze the essential stability of Nash equilibria for games with finitely many players.

As we describe in the introduction, our goal is to contribute to this growing literature by addressing essential stability properties of Cournot-Nash equilibria in large generalized games. However, results of essential stability for games with finitely many players take advantage of the fact that the equilibrium correspondence[3] has a closed graph, with non-empty and compact values. In fact, with these properties, the equilibrium correspondence is generically lower hemicontinuous, which in turn implies essential stability as a generic property.

In our case, under mild conditions on the features of a large generalized game, a pure strategy Cournot-Nash equilibrium always exists.[4] However, the equilibrium correspondence may not have compact values (see footnote 8), and therefore, the traditional analysis of essential stability cannot be implemented directly in our context. Nevertheless, associated with any Cournot-Nash equilibrium of a large generalized game, there is a vector of messages (generated by strategy profiles of non-atomic players) and a vector of optimal strategies of atomic players. These message–action vectors constitute all the relevant information that a player takes into account to make optimal

[3] That is, the correspondence that associates games with the set of its pure strategy equilibria.

[4] See Schmeidler (1973) and Rath (1992) for continuous large games, Balder (1999, 2002) for continuous large generalized games, and Carmona and Podczeck (2014) for discontinuous large generalized games.

decisions. In addition, the correspondence that associates games with the set of equilibrium messages and atomic players' profiles has closed graph and compact values (see Theorem 1 and 5). Hence, we focus our analysis on the stability of equilibrium messages–actions with respect to perturbations in the game.

3 Continuous large generalized games

Through our model, some characteristics of large generalized games are fixed and summarized by a tuple $(T_1, T_2, \widehat{K}, (\widehat{K}_t)_{t\in T_2}, H)$, where $T_1 \cup T_2$ is a non-empty set. T_1 is a compact subset of a metric space and represents the family of non-atomic players. There is a σ-algebra $\mathcal{A}$ and a finite measure μ such that $(T_1, \mathcal{A}, \mu)$ is a complete atomless measure space. The set of atomic players is represented by the finite set T_2. $\widehat{K}$ is the non-empty and compact metric space where non-atomic players' strategies belong. Each atomic player $t \in T_2$ has strategies in $\widehat{K}_t$, which is a non-empty and compact subset of a normed vector space equipped with a metric induced by a norm. The function $H : T_1 \times \widehat{K} \to \mathbb{R}^m$ codifies non-atomic players' strategies, and it is continuous with respect to the product topology induced by the metrics of T_1 and $\widehat{K}$.

Let $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ be a large generalized game where each player $t \in T_1 \cup T_2$ is characterized by a tuple (K_t, Γ_t, u_t). A non-atomic player $t \in T_1$ has a non-empty and closed action space $K_t \subseteq \widehat{K}$, while each atomic player $t \in T_2$ has a non-empty, closed, and convex action space $K_t \subseteq \widehat{K}_t$. We assume that the correspondence $t \in T_1 \twoheadrightarrow K_t$ is measurable, i.e., $\{t \in T_1 : K_t \cap F \neq \emptyset\} \in \mathcal{A}$ for each closed subset F of $\widehat{K}$.

A strategy profile for non-atomic players is given by a function $f : T_1 \to \widehat{K}$ such that $f(t) \in K_t$, for any $t \in T_1$. Any vector $a = (a_t)_{t\in T_2} \in \prod_{t\in T_2} K_t$ is a strategy profile for atomic players. Given $i \in \{1, 2\}$, let $\mathcal{F}^i((K_t)_{t\in T_i})$ be the space of strategy profiles for agents in T_i. In addition, for any $t \in T_2$, let $\mathcal{F}^2_{-t}((K_s)_{s\in T_2\setminus\{t\}}) := \prod_{s\in T_2\setminus\{t\}} K_s$ be the set of strategy profiles for agents in $T_2\setminus\{t\}$.

Each participant to the game considers aggregated information about strategies of non-atomic players, that is, if non-atomic players choose a strategy profile $f \in \mathcal{F}^1((K_t)_{t\in T_1})$, then the relevant characteristics of this profile are coded by the function H, and each player takes into account aggregated information about these available characteristics through the message $m(f) := \int_{T_1} H(t, f(t))d\mu$. In other words, $m(f)$ is what the players know about the strategic choices of non-atomic players.

We concentrate our attention only on strategy profiles for which messages are well defined. Thus, let $\mathcal{F}^1_m((K_t)_{t\in T_1})$ be the set of profiles $f \in \mathcal{F}^1((K_t)_{t\in T_1})$ such that $H(\cdot, f(\cdot)) : T_1 \to \mathbb{R}^m$ is measurable.[5] Therefore, the set of messages associated with non-atomic players' strategies is given by

$$M((K_t)_{t\in T_1}) = \left\{ \int_{T_1} H(t, f(t))d\mu : \quad f \in \mathcal{F}^1_m((K_t)_{t\in T_1}) \right\}.$$

[5] That is, for any Borelian set $E \subseteq \mathbb{R}^m$, $\{t \in T_1 : H(t, f(t)) \in E\}$ belongs to $\mathcal{A}$.

Let $\widehat{M} = M((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^1 = \mathcal{F}^1((\widehat{K})_{t\in T_1})$, $\widehat{\mathcal{F}}^2 = \mathcal{F}^2((\widehat{K}_t)_{t\in T_2})$, and $\widehat{\mathcal{F}}^2_{-t} = \mathcal{F}^2_{-t}((\widehat{K}_s)_{s\in T_2\setminus\{t\}})$.

Messages and strategy profiles may restrict players' admissible strategies, that is, the set of strategies available for a player $t \in T_1$ is determined by a correspondence $\Gamma_t : \widehat{M}\times\widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ with non-empty and compact values, where for every $(m, a) \in \widehat{M}\times\widehat{\mathcal{F}}^2$, the correspondence $t \in T_1 \twoheadrightarrow \Gamma_t(m, a)$ is measurable, i.e., $\{t \in T_1 : \Gamma_t(m, a) \cap F \neq \emptyset\} \in \mathcal{A}$ for each closed subset F of $\widehat{K}$. Analogously, feasible strategies for a player $t \in T_2$ are determined by a correspondence $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2_{-t} \twoheadrightarrow K_t$ with non-empty, compact and convex values.

Each $t \in T_1$ has an objective function $u_t : \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{R}$ such that, for every $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the function $(t, x) \in T_1 \times \widehat{K} \to u_t(x, m, a)$ is $\mathcal{A} \times \mathcal{B}(\widehat{K})$-measurable, where $\mathcal{B}(\widehat{K})$ refers to the Borel σ-algebra of $\widehat{K}$ and $\mathcal{A}\times\mathcal{B}(\widehat{K})$ denotes the product σ-algebra. Also, the map $U : T_1\times\widehat{K}\times\widehat{M}\times\widehat{\mathcal{F}}^2 \to \mathbb{R}$ given by $U(t, x, m, a) = u_t(x, m, a)$ is bounded. Each atomic player $t \in T_2$ has a bounded objective function $u_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{R}$, which is quasi-concave on its own strategy a_t.

Definition 1 (*Cournot-Nash Equilibrium*) A Cournot-Nash equilibrium of a large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ is given by strategy profiles $(f^*, a^*) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$, with $m(f^*) \in \widehat{M}$, such that

(i) For almost all $t \in T_1$, $f^*(t) \in \Gamma_t(m^*, a^*)$, where $m^* = m(f^*)$, and

$$u_t(f^*(t), m^*, a^*) \geq u_t(f(t), m^*, a^*), \quad \forall f(t) \in \Gamma_t(m^*, a^*).$$

(ii) For any $t \in T_2$, $a^*_t \in \Gamma_t(m^*, a^*_{-t})$ and

$$u_t(m^*, a^*_t, a^*_{-t}) \geq u_t(m^*, a_t, a^*_{-t}), \quad \forall a_t \in \Gamma_t(m^*, a^*_{-t}).$$

Assume that the following conditions hold:

(A1) *For any $t \in T_1 \cup T_2$, the objective function u_t is continuous.*
(A2) *For any $t \in T_1 \cup T_2$, the correspondence Γ_t is continuous.*[6]

Taking as given $(T_1, T_2, \widehat{K}, (\widehat{K}_t)_{t\in T_2}, H)$, let $\mathbb{G}$ be the collection of large generalized games satisfying the conditions described above, i.e., the set of continuous large generalized games where atomic players have quasi-concave objective functions and convex sets of admissible strategies. It follows from Balder (2002, Theorem 2.2.1) that any $\mathcal{G} \in \mathbb{G}$ has a non-empty set of Cournot-Nash equilibria, denoted by $\mathrm{CN}(\mathcal{G})$.[7]

[6] Given $t \in T_1$, continuity of $\Gamma_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow K_t$ requires that it be both upper hemicontinuous and lower hemicontinuous. Upper hemicontinuity is satisfied at $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ when for any open set $A \subseteq K_t$ such that $\Gamma_t(m, a) \subseteq A$, there is an open neighborhood $U \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ of (m, a) such that $\Gamma_t(m', a') \subseteq A$ for every $(m', a') \in U$. Lower hemicontinuity is satisfied at $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ when for any open set $A \subseteq K_t$ such that $\Gamma_t(m, a) \cap A \neq \emptyset$, there is an open neighborhood $U \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ of (m, a) such that $\Gamma_t(m', a') \cap A \neq \emptyset$ for every $(m', a') \in U$. Same definitions apply for the correspondences of admissible strategies associated with atomic players $(\Gamma_t)_{t\in T_2}$.

[7] Under our assumptions, for every $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ the correspondences $t \in T_1 \twoheadrightarrow K_t$ and $t \in T_1 \twoheadrightarrow \Gamma_t(m, a)$ have measurable graph [see Aliprantis and Border (2006, Lemma 18.2 and Theorem 18.6)].

We endow the set $\mathbb{G}$ with the following metric:

$$\begin{aligned}\rho(\mathcal{G}_1, \mathcal{G}_2) &= \sup_{t \in T_1} \sup_{(x,m,a) \in \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2} |u_t^1(x, m, a) - u_t^2(x, m, a)| \\ &+ \sup_{t \in T_1} \sup_{(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2} d_H \left(\Gamma_t^1(m, a), \Gamma_t^2(m, a)\right) + \sup_{t \in T_1} d_H \left(K_t^1, K_t^2\right) \\ &+ \max_{t \in T_2} \sup_{(m,x,a_{-t}) \in \widehat{M} \times \widehat{K}_t \times \widehat{\mathcal{F}}^2_{-t}} |u_t^1(m, x, a_{-t}) - u_t^2(m, x, a_{-t})| \\ &+ \max_{t \in T_2} \sup_{(m,a_{-t}) \in \widehat{M} \times \widehat{\mathcal{F}}^2_{-t}} d_{H,t} \left(\Gamma_t^1(m, a_{-t}), \Gamma_t^2(m, a_{-t})\right) \\ &+ \max_{t \in T_2} d_{H,t} \left(K_t^1, K_t^2\right),\end{aligned}$$

where $\mathcal{G}_i = \mathcal{G}_i((K_t^i, \Gamma_t^i, u_t^i)_{t \in T_1 \cup T_2})$, d_H denotes the Hausdorff distance induced by the metric of $\widehat{K}$ over the collection of its non-empty and compact subsets and, for every $t \in T_2$, the Hausdorff distance induced by the metric of $\widehat{K}_t$ is denoted by $d_{H,t}$.

Proposition 1 *The space of continuous large generalized games* $(\mathbb{G}, \rho)$ *is complete.*

The proof is given in the Appendix.

We point out that the correspondence associating the parameters that define a generalized game with the set of its Cournot-Nash equilibria is not necessarily compact-valued,[8] a property that was required by the previous literature on essential stability in games with finitely many players. However, given any Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, the pair $(m(f^*), a^*)$ contains all the information that players take into account to make their decisions. Thus, we focus our analysis of stability on the effects that perturbations on the characteristics of a game have on both messages from non-atomic players and strategies of atomic players that are consistent with a Cournot-Nash equilibrium.[9]

Footnote 7 continued
On the other hand, Balder's result requires that players have a common universal action space. However, his result can be extended allowing different universal action spaces for atomic and non-atomic players [see Balder (2002), page 448, remark (v)].

[8] For instance, consider an electoral game with a continuum of non-atomic players $T_1 = [0, 1]$, which vote for a party in $\{a, b\}$. Let x_t be the action of player $t \in T_1$ and assume that his objective function u_t only takes into account the benefits that he receives from the election of parties, given by $\{v_t(a), v_t(b)\}$, weighted by the support that each party has in the population, i.e., $u_t \equiv v_t(a)\mu(\{s \in T_1 : x_s = a\}) + v_t(b)(1 - \mu(\{s \in T_1 : x_s = a\}))$, where μ denotes the Lebesgue measure in $[0, 1]$, that is, the utility level of a player $t \in T_1$ in unaffected by his own action and, therefore, any measurable profile $x : [0, 1] \to \{a, b\}$ constitutes a Nash equilibrium of the game. Hence, the set of Nash equilibria is not compact. However, if we consider that each player receives as a message the support that party a has in the population, $m = \mu(\{s \in T_1 : x_s = a\})$, then the set of equilibrium messages is equal to $[0, 1]$, which is a compact set.

[9] Since action profiles are coded using the function H, there may exist several Cournot-Nash equilibria inducing a same message. Even that, this indetermination does not have real effects on players utility levels.

Definition 2 (*The Cournot-Nash Correspondence*) The Cournot-Nash correspondence of $\mathbb{G}$ is given by the multivalued mapping $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ that associates to any $\mathcal{G} \in \mathbb{G}$ the set of messages and actions $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ such that for some $f^* \in \widehat{\mathcal{F}}^1$ we have $m^* = m(f^*)$ and $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$.

4 Essential stability of equilibria for continuous games

We analyze how the set of Cournot-Nash equilibria of a large generalized game in $\mathbb{G}$ changes when the characteristics of players are modified. Our analysis is based on the concept of *essential stability* introduced by Fort (1950) for fixed points of single-valued mappings and by Jiang (1962) for correspondences.

Definition 3 (*Essential Equilibrium*) Let $\mathbb{G}' \subseteq \mathbb{G}$. Given $\mathcal{G}_0 \in \mathbb{G}'$, $(f^*, a^*) \in \mathrm{CN}(\mathcal{G}_0)$ is an essential equilibrium of $\mathcal{G}_0$ with respect to $\mathbb{G}'$ when, for any open neighborhood $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ of $(m(f^*), a^*)$ there exists $\epsilon > 0$ such that $\Lambda(\mathcal{G}) \cap O \neq \emptyset$ for any $\mathcal{G} \in \mathbb{G}'$ satisfying $\rho(\mathcal{G}_0, \mathcal{G}) < \epsilon$. The large generalized game $\mathcal{G}_0$ is essential with respect to $\mathbb{G}'$ if all its Cournot-Nash equilibria are essential with respect to $\mathbb{G}'$.

Hence, a large generalized game $\mathcal{G}_0 \in \mathbb{G}'$ is essential with respect to $\mathbb{G}' \subseteq \mathbb{G}$ if and only if messages and atomic players' strategies associated with a Cournot-Nash equilibrium of $\mathcal{G}_0$ can be approximated by equilibrium messages and strategies of generalized games in $\mathbb{G}'$ close to $\mathcal{G}_0$. Note that if $\mathcal{G}_0$ is essential with respect to $\mathbb{G}'$, then it is essential with respect to any non-empty set $\mathbb{G}'' \subseteq \mathbb{G}'$ such that $\mathcal{G}_0 \in \mathbb{G}''$. Unfortunately, as the following example shows, not all games in $\mathbb{G}$ are essential.

Example 1 Suppose that $T_1 = [0, 1], T_2 = \{\alpha\}, \widehat{K} = \{0, 1\}$ and $\widehat{K}_\alpha = [0, 1]$. Consider a generalized game $\mathcal{G}$ where for each $t \in T_1, (K_t, \Gamma_t) \equiv (\widehat{K}, \widehat{K})$, $(K_\alpha, \Gamma_\alpha) \equiv (\widehat{K}_\alpha, \widehat{K}_\alpha)$ and $H(\cdot, x) \equiv x$. In addition, $u_\alpha(m, x) = -|m - x|^2$ and $(u_t(0, m, a_\alpha), u_t(1, m, a_\alpha)) = (1, 1), \forall t \in T_1$. Note that, $\mathcal{G}$ has a continuum of Cournot-Nash equilibria and $\Lambda(\mathcal{G}) = \{(\lambda, \lambda) \in \mathbb{R}^2 : \lambda \in [0, 1]\}$.

Given $\lambda \in [0, 1]$ and $\epsilon > 0$, let $\mathcal{G}_{\lambda,\epsilon}$ be the game obtaining from $\mathcal{G}$ by changing the objective functions of non-atomic players to

$$(u_t^{\lambda,\epsilon}(0, m, a_\alpha), u_t^{\lambda,\epsilon}(1, m, a_\alpha)) = \begin{cases} (1+\epsilon,\ 1), & \text{for any } t \in [0, 1-\lambda] \\ (1,\ 1+\epsilon), & \text{for any } t \in (1-\lambda, 1]. \end{cases}$$

It follows that $\mathcal{G}_{\lambda,\epsilon}$ has only one Cournot-Nash equilibrium and $\Lambda(\mathcal{G}_{\lambda,\epsilon}) = \{(\lambda, \lambda)\}$. Since $(\lambda, \epsilon) \in [0, 1] \times \mathbb{R}_{++}$ is arbitrary and $\rho(\mathcal{G}, \mathcal{G}_{\lambda,\epsilon}) \leq \epsilon$, we conclude that $\mathcal{G}$ has no essential Cournot-Nash equilibrium with respect to $\mathbb{G}$. □

Theorem 1 *Given a closed set $\mathbb{G}' \subseteq \mathbb{G}$, the collection of large generalized games that are essential with respect to $\mathbb{G}'$ is a dense residual subset of $\mathbb{G}'$.*[10] *For any $\mathcal{G} \in \mathbb{G}'$, if $\Lambda(\mathcal{G})$ is a singleton, then $\mathcal{G}$ is essential with respect to $\mathbb{G}'$.*

[10] A subset of $\mathbb{G}'$ is residual if it contains the intersection of a countable family of dense and open subsets of $\mathbb{G}'$.

The proof is given in the Appendix.

It follows from Theorem 1 that given $\mathcal{G}_0 \in \mathbb{G}'$, for any $\epsilon > 0$, an essential generalized game $\mathcal{G} \in \mathbb{G}'$ exists such that $\rho(\mathcal{G}_0, \mathcal{G}) < \epsilon$.

The next example applies our results to show that equilibrium messages and atomic players strategies of an essential large game can be approximated by equilibrium messages and atomic player strategies of *finite action* large games.[11]

Example 2 Let $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2}) \in \mathbb{G}$ be an essential game with respect to $\mathbb{G}$, with $(K_t, K_s) \equiv (\Gamma_t, \Gamma_s) \equiv (\widehat{K}, \widehat{K_s})$, $\forall (t, s) \in T_1 \times T_2$. The compactness of strategy sets ensures that for each $t \in T_1 \cup T_2$, there is a countable and dense subset $\{x_{t,n} : n \in \mathbb{N}\}$ of K_t. Thus, given $n \in \mathbb{N}$, let $\mathcal{G}_n = \mathcal{G}_n((K_t^n, \Gamma_t^n, u_t)_{t \in T_1 \cup T_2}) \in \mathbb{G}$ be the finite action generalized game characterized by $\Gamma_t^n \equiv K_t^n := \{x_{t,1}, \ldots, x_{t,n}\}$. It follows that $\rho(\mathcal{G}, \mathcal{G}_n)$ converges to zero as n goes to infinity.

As $\mathcal{G}$ is essential with respect to $\mathbb{G}$, it follows from Definition 3 that given messages and atomic players' strategies $(m, a) \in \Lambda(\mathcal{G})$, for each $\delta > 0$ there is $n(\delta) \in \mathbb{N}$ such that, given $n > n(\delta)$ some $(m_n, a_n) \in \Lambda(\mathcal{G}_n)$ is δ-close to (m, a). □

Although there are games that do not have essential equilibria, any $\mathcal{G} \in \mathbb{G}$ has subsets of Cournot-Nash equilibria that are stable. To formalize this property, we extend the concept of essential stability to subsets of equilibrium points.

Definition 4 (*Essential Set*) Let $\mathbb{G}' \subseteq \mathbb{G}$. Given $\mathcal{G}_0 \in \mathbb{G}'$, a subset $e(\mathcal{G}_0) \subseteq \Lambda(\mathcal{G}_0)$ is essential with respect to $\mathbb{G}'$ if it is non-empty, compact, and for any open set $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$,

$$[e(\mathcal{G}_0) \subseteq O] \Longrightarrow \left[\exists \epsilon > 0 : \mathcal{G} \in \mathbb{G}', \;\; \rho(\mathcal{G}_0, \mathcal{G}) < \epsilon \Longrightarrow \Lambda(\mathcal{G}) \cap O \neq \emptyset\right].$$

A minimal essential set with respect to $\mathbb{G}'$ is a minimal element ordered by set inclusion in the family of essential subsets of $\Lambda(\mathcal{G}_0)$ with respect to $\mathbb{G}'$. A component of $\Lambda(\mathcal{G}_0)$ is a maximal connected subset of $\Lambda(\mathcal{G}_0)$ ordered by set inclusion.

Definition 4 adapts to our framework the concepts of essential sets and components that were introduced by Jiang (1963) and Yu and Yang (2004) in the context of stability of fixed point of multivalued mappings. These concepts were also addressed by Zhou et al. (2007) to study the stability of mixed-strategy equilibria in non-convex finite-player games.

Since the Cournot-Nash correspondence Λ is upper hemicontinuous with non-empty and compact values (see the proof of Theorem 1), for any $\mathcal{G} \in \mathbb{G}' \subseteq \mathbb{G}$, the set $\Lambda(\mathcal{G})$ is essential with respect to $\mathbb{G}'$. Moreover, given $A \subset B \subseteq \Lambda(\mathcal{G})$, if A is essential with respect to $\mathbb{G}'$ and B is compact, then B is essential with respect to $\mathbb{G}'$ too.[12] Thus, we focus the attention on the existence of minimal essential sets.

Let $\mathbb{G}' \subseteq \mathbb{G}$ be a closed set. Some results can be inferred from Theorem 1:

[11] For general results of strategic approximations of continuous games by finite games, see Reny (2011).

[12] Indeed, B is non-empty and compact. Also, for any open set $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $B \subseteq O$, we have that $A \subset O$. Thus, the essentiality of A with respect to $\mathbb{G}'$ ensures that B is essential too.

(i) If for some $\mathcal{G} \in \mathbb{G}'$ there is an essential Cournot-Nash equilibrium $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$, then $\{(m(f^*), a^*)\}$ is a minimal essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$. Therefore, it follows from Theorem 1 that *any closed set $\mathbb{G}' \subseteq \mathbb{G}$ has a dense residual subset in which any game has at least one minimal essential subset with respect to $\mathbb{G}'$ that is also connected.*

(ii) Since for any $\mathcal{G} \in \mathbb{G}'$ the set $\Lambda(\mathcal{G})$ is compact, any component of $\Lambda(\mathcal{G})$ is non-empty, connected, and compact.[13] Hence, when $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$ is essential with respect to $\mathbb{G}'$, the component associated with $\{(m(f^*), a^*)\}$ is an essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$ (because it is compact and contains the essential set $\{(m(f^*), a^*)\}$). Therefore, it follows from Theorem 1 that *any closed set $\mathbb{G}' \subseteq \mathbb{G}$ has a dense residual subset in which any game has at least one essential component.*

The following result extends the two properties above ensuring that they hold for *every* large generalized game when sets of strategies are subsets of normed spaces.

Theorem 2 *Given a closed set $\mathbb{G}' \subseteq \mathbb{G}$, for each $\mathcal{G} \in \mathbb{G}'$ the following properties hold:*

(i) *A minimal essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$ always exists.*

(ii) *If $\Lambda(\mathcal{G})$ has a connected essential set with respect to $\mathbb{G}'$, then it has an essential component.*

(iii) *If $\widehat{K}$ is a convex subset of a normed space equipped with a metric induced by a norm, then every minimal essential set of $\Lambda(\mathcal{G})$ is connected. Furthermore, if $\Lambda(\mathcal{G})$ is finite, then at least one Cournot-Nash equilibrium of $\mathcal{G}$ is essential with respect to $\mathbb{G}'$.*

The proof is given in the Appendix.

It is natural to ask whether the continuity of payoff functions is necessary to ensure the stability properties previously discussed. The following example points out that if players' objective functions are discontinuous, then the metric space of large generalized games with a non-empty set of Cournot-Nash equilibria becomes incomplete. Hence, the results of Fort (1950) and Jiang (1962) cannot be adapted to our framework, compromising the validity of Theorems 1 and 2.

Example 3 Suppose that $T_1 = [0, 1]$, $T_2 = \emptyset$, $\widehat{K} = [0, 1]$ and $H(t, x) = x$. Thus, $\widehat{M} = [0, 1]$.

For any $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the large generalized game where each $t \in T_1$ is characterized by $K_t^n = \left[\frac{1}{n}, 1\right]$, $\Gamma_t^n \equiv K_t^n$ and $u_t^n = u$, where $u : [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$u(x, m) = \begin{cases} 2 & \text{when } m \neq 0 \wedge x = 0; \\ x & \text{in other case.} \end{cases}$$

[13] By definition, components are non-empty. Since a component is a union of connected sets with at least one common element, it is connected too. Since the closure of a connected set is connected, components of compact sets are closed and, therefore, compact [for more details, see Berge (1997, page 98)].

The game $\mathcal{G}_n$ has a unique Cournot-Nash equilibrium, because every player t has the same dominant strategy $f^*(t) = 1$. Therefore, $\Lambda(\mathcal{G}_n) = \{1\}$.

Let $\overline{\mathcal{G}}$ be the discontinuous large generalized game where each player $t \in T_1$ is characterized by $K_t = [0, 1]$, $\Gamma_t(m) = K_t$ and $u_t = u$. Note that $\overline{\mathcal{G}}$ has an empty set of Cournot-Nash equilibria. Indeed, when $m = 0$, the optimal strategy of every player t is $f(t) = 1$, inducing the message $m' = 1$. Otherwise, when $m \neq 0$, every player t chooses $f(t) = 0$, inducing the message $m' = 0$.

Since $\lim_n \rho(\mathcal{G}_n, \overline{\mathcal{G}}) = 0$, the metric space of discontinuous large generalized games that have equilibria is incomplete. □

To obtain the result of Example 3, it is crucial to allow perturbations on action sets. Indeed, when action sets and correspondences of admissible strategies are fixed, it is possible to extend our analysis to some spaces of discontinuous games (see Sect. 7).

5 Essential stability for the parameterizations of the space $\mathbb{G}$

In this section, we discuss the stability of Cournot-Nash equilibria when only some characteristics of the large generalized game are perturbed.

Definition 5 (*Parametrization*) A parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}$ is given by a complete metric space of parameters $(\mathbb{X}, \tau)$ and a continuous function $\kappa : \mathbb{X} \to \mathbb{G}$ that associates parameters with generalized games.

Definition 6 (*$\mathcal{T}$-Essential Equilibrium*) Let $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ be a parametrization of $\mathbb{G}$. Given $\mathcal{X}_0 \in \mathbb{X}$, a Cournot-Nash equilibrium $(f^*, a^*) \in \text{CN}(\kappa(\mathcal{X}_0))$ is essential with respect to $\mathbb{X}$ under κ, if for any open neighborhood $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ of $(m(f^*), a^*)$, there exists $\epsilon > 0$ such that $\Lambda(\kappa(\mathcal{X})) \cap O \neq \emptyset$ for any parameter $\mathcal{X} \in \mathbb{X}$ satisfying $\tau(\mathcal{X}_0, \mathcal{X}) < \epsilon$. A generalized game $\mathcal{G}_0 \in \mathbb{G}$ is $\mathcal{T}$-essential if there exists $\mathcal{X}_0 \in \mathbb{X}$ such that $\mathcal{G}_0 = \kappa(\mathcal{X}_0)$ and all its Cournot-Nash equilibria are essential with respect to $\mathbb{X}$ under κ.

Definition 7 (*$\mathcal{T}$-Essential Set*) Given $\mathcal{G}_0 \in \mathbb{G}$, a subset $e(\mathcal{G}_0) \subseteq \Lambda(\mathcal{G}_0)$ is $\mathcal{T}$-essential—or essential with respect to $\mathbb{X}$ under κ—if there exists a parameter $\mathcal{X}_0 \in \mathbb{X}$ such that (i) $\mathcal{G}_0 = \kappa(\mathcal{X}_0)$; (ii) $e(\mathcal{G}_0)$ is non-empty and compact; and (iii) for any open set $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ with $e(\mathcal{G}_0) \subseteq O$ there exists $\epsilon > 0$ such that, if $\mathcal{X} \in \mathbb{X}$ and $\tau(\mathcal{X}_0, \mathcal{X}) < \epsilon$, then $\Lambda(\kappa(\mathcal{X})) \cap O \neq \emptyset$.

Some remarks:

(i) Since $\kappa : \mathbb{X} \to \mathbb{G}$ is (τ, ρ)-continuous, $\mathcal{G}$ is essential with respect to $\mathbb{G}' \subseteq \mathbb{G}$ if and only if it is $\mathcal{T}$-essential for any parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ such that $\mathcal{G} = \kappa(\mathcal{X})$ for some $\mathcal{X} \in \mathbb{X}$.

(ii) Assume that $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ satisfies $\mathbb{X} \subseteq \mathbb{G}$, $\tau = \rho$, and κ is the immersion of $\mathbb{X}$ on $\mathbb{G}$. Then, for any $\mathcal{X} \in \mathbb{X}$, $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential if and only if $\mathcal{X}$ is essential with respect to $\mathbb{X}$.

The following result states stability properties of Cournot-Nash equilibria when perturbations are determined by a parametrization of $\mathbb{G}$. Hence, we extend Theorems

1 and 2, obtaining stability results of Cournot-Nash equilibria when some but not necessarily all characteristics that define a generalized game are allowed to change.[14]

Theorem 3 *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ of $\mathbb{G}$, the collection of parameters $\mathcal{X} \in \mathbb{X}$ for which $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential is a dense residual subset of $\mathbb{X}$. Furthermore, for any $\mathcal{X} \in \mathbb{X}$, we have that*

(i) *If $\Lambda(\kappa(\mathcal{X}))$ is a singleton, then $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential.*
(ii) *There is a minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$.*
(iii) *Any $\mathcal{T}$-essential and connected set $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$ is contained in a $\mathcal{T}$-essential component.*
(iv) *Suppose that $\mathbb{X}$ is a convex subset of a normed space and τ is a metric induced by a norm. Then, every minimal $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is connected.*

Proof The proof of Theorem 1 ensures that Λ has closed graph with non-empty and compact values. Since $\kappa\mathbb{X} \to \mathbb{G}$ is continuous and $(\mathbb{X}, \tau)$ is a complete metric space, the same properties hold for the correspondence $\Lambda \circ \kappa : \mathbb{X} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$. Hence, the first two properties follow from identical arguments to those made in the proof of Theorem 1. Also, (ii)–(iv) can be obtained by analogous arguments to those made in the proof of Theorem 2, changing $(\mathbb{G}, \rho, \Lambda)$ by $(\mathbb{X}, \tau, \Lambda \circ \kappa)$. □

The following example shows that the continuity requirement on the definition of a parametrization $((\mathbb{X}, \tau), \kappa)$ cannot be relaxed without compromising the results of Theorem 3.

Example 4 Assume that $T_1 = [0, 1]$ and $\widehat{K}$ is a convex subset of a normed space with a metric induced by a norm. Fix $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{G}$ such that $\Lambda(\mathcal{G}_1) \cap \Lambda(\mathcal{G}_2) = \emptyset$, and consider a tuple $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ with $\mathbb{X} = [0, 1]$, $\tau(x_1, x_2) = |x_1 - x_2|$, and $\kappa(\mathcal{X}) = a(\mathcal{X})\mathcal{G}_1 + (1 - a(\mathcal{X}))\mathcal{G}_2$, $\forall t \in [0, 1]$, where $a : [0, 1] \to \{0, 1\}$ satisfies $a(t) = 0$ if and only if t is a rational number.[15]

Note that, as $\Lambda(\mathcal{G}_1)$ and $\Lambda(\mathcal{G}_2)$ are disjoint compact sets, there are disjoint open sets $O_1, O_2 \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $\Lambda(\mathcal{G}_i) \subset O_i$, $\forall i \in \{1, 2\}$. Thus, it follows from Definition 6 that the collection of parameters $\mathcal{X} \in \mathbb{X}$ for which $\kappa(\mathcal{X})$ is $\mathcal{T}$-essential is an empty set. □

Following the ideas of Yu, Yang, and Xiang (2005, Theorems 4.1 and 4.2), we end this section with results about the stability of essential sets and components.

[14] For instance, when there are personalized perturbations on players' characteristics, as an example, fix a game $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2}) \in \mathbb{G}$. Given $i \in \{1, 2\}$, let $T_i^a, T_i^b, T_i^c \subseteq T_i$ be, respectively, the subsets of players in T_i for which we allow perturbations on objective functions, on strategy sets, and on the correspondences of admissible strategies. Let $\mathbb{G}_{\mathcal{G}} \subseteq \mathbb{G}$ be the set of generalized games $\mathcal{G}((\widetilde{K}_t, \widetilde{\Gamma}_t, \widetilde{u}_t)_{t\in T_1\cup T_2})$ such that (1) for any $t \in (T_1\backslash T_1^a) \cup (T_2\backslash T_2^a)$, $\widetilde{u}_t = u_t$; (2) for any $t \in (T_1\backslash T_1^b) \cup (T_2\backslash T_2^b)$, $\widetilde{K}_t = K_t$; and (3) for any $t \in (T_1\backslash T_1^c) \cup (T_2\backslash T_2^b)$, $\widetilde{\Gamma}_t = \Gamma_t$. Since $\mathbb{G}_{\mathcal{G}}$ is ρ-closed, it follows that $(\mathbb{G}_{\mathcal{G}}, \rho)$ is a complete metric space. Therefore, since the immersion $\iota : \mathbb{G}_{\mathcal{G}} \hookrightarrow \mathbb{G}$ is continuous, $((\mathbb{G}_{\mathcal{G}}, \rho), \iota)$ is a parametrization of $\mathbb{G}$.

[15] Suppose that $\mathcal{G}_1 = \mathcal{G}_1((K_t^1, \Gamma_t^1, u_t^1)_{t\in T_1\cup T_2})$ and $\mathcal{G}_2 = \mathcal{G}_2((K_t^2, \Gamma_t^2, u_t^2)_{t\in T_1\cup T_2})$. Since $\widehat{K}$ and $\widehat{K}_t$, where $t \in T_2$, are convex subsets of normed spaces with metrics induced by norms, for each $\lambda \in [0, 1]$, the convex combination $\lambda\mathcal{G}_1 + (1-\lambda)\mathcal{G}_2$ is well defined and given by the game $\mathcal{G}((\lambda K_t^1 + (1-\lambda)K_t^2, \lambda\Gamma_t^1 + (1-\lambda)\Gamma_t^2, \lambda u_t^1 + (1-\lambda)u_t^2)_{t\in T_1\cup T_2})$. Recall that, given subsets A and B of a vectorial space, $\lambda A + (1-\lambda)B := \{\lambda a + (1-\lambda)b : (a, b) \in A \times B\}$.

Definition 8 (*Stability of Sets*) Fix a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ and $\mathcal{X} \in \mathbb{X}$.

(i) The set $E \subseteq \Lambda(\kappa(\mathcal{X}))$ is stable if for every $\epsilon > 0$, there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$, there exists a minimal $\mathcal{T}$-essential set $E' \subseteq \Lambda(\kappa(\mathcal{X}'))$ for which

$$E' \subseteq B[\epsilon, E] := \{(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2 : \exists (m', a') \in E,\ \widehat{\sigma}((m, a), (m', a')) \leq \epsilon\},$$

where $\widehat{\sigma}$ is the metric associated with the product topology of $\mathbb{R}^m \times \widehat{\mathcal{F}}^2$.

(ii) The set $E \subseteq \Lambda(\kappa(\mathcal{X}))$ is strongly stable if for every $\epsilon > 0$, there is $\delta > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$, there exists a $\mathcal{T}$-essential component $E' \subseteq \Lambda(\kappa(\mathcal{X}'))$ for which $E' \subseteq B[\epsilon, E]$.

Note that, if $E \in \Lambda(\kappa(\mathcal{X}))$ is strongly stable, then it is stable.[16] Also, any subset of $\Lambda(\kappa(\mathcal{X}))$ which contains a (strongly) stable set is (strongly) stable too.

Theorem 4 *Given a parametrization $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ and $\mathcal{X} \in \mathbb{X}$, the $\mathcal{T}$-essential subsets of $\Lambda(\kappa(\mathcal{X}))$ are stable. Furthermore, if $\mathbb{X}$ is a convex subset of a normed space and τ is induced by a norm, then the $\mathcal{T}$-essential components of $\Lambda(\kappa(\mathcal{X}))$ are strongly stable.*

The proof is given in the Appendix.

6 Essential stability as a rationale for electoral participation

In a recent paper, Barlo and Carmona (2011) introduced the refinement concept of *strategic equilibria* in large games. Intuitively, a Nash equilibrium of a large game is strategic if it is the limit of equilibria of abstract perturbed games, where players believe that they have a positive impact on the social choice.[17] As an application of their results, they give a rationale to explain why electors vote for their favorite candidate. Introducing a large game with proportional voting, they show that there is a continuum of Cournot-Nash equilibria, but only one strategic equilibrium: that in which electors vote by their favorite party [see Barlo and Carmona (2011, Example 2.1)].

Inspired by this result, we analyze a large generalized electoral game where electors have different degrees of political interest. The Cournot-Nash equilibrium where only politically engaged players vote and support their favorite party appears as the unique $\mathcal{T}$-essential equilibrium of our electoral game, for some parametrization $\mathcal{T}$.

[16] It is sufficient to prove that any $\mathcal{T}$-essential component contains a minimal $\mathcal{T}$-essential set. Fix an $\mathcal{T}$-essential component $C \subseteq \Lambda(\kappa(\mathcal{X}))$. Let $\mathcal{S}_C$ be the family of $\mathcal{T}$-essential subsets of $\Lambda(\kappa(\mathcal{X}))$ contained in C, endowed with the partial order determined by set inclusion. Since essential sets are non-empty and compact, any totally ordered subset of $\mathcal{S}_C$ has a lower bound. By Zorn's Lemma, $\mathcal{S}_C$ has a minimal element, which concludes the proof.

[17] More precisely, given a large game $\mathcal{G}$ with only non-atomic players, for any $\epsilon > 0$ define an ϵ-perturbed game $\mathcal{G}_\epsilon$ where every player perceives that he, but no other, has a positive small impact on the social choice. Then, following our notation, $(f, a) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ is a *strategic equilibrium* for a game $\mathcal{G}$ if there exists $\{\epsilon_k\}_{k\in\mathbb{N}} \subset (0, 1)$ decreasing to zero, and a sequence $\{(f_k, a_k)\}_{k\in\mathbb{N}} \subset \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ converging to (f, a), such that (f_k, a_k) is a Cournot-Nash equilibrium for $\mathcal{G}_{\epsilon_k}$ for any $k \in \mathbb{N}$.

Given a set of parties $P = \{1, \ldots, \overline{p}\}$ and a parameter $\mu \geq 0$, consider an electoral game $\mathcal{E}_\alpha = \mathcal{E}_\alpha(T_1, T_2, \widehat{K}, (\widehat{K}_t)_{t \in T_2}, H, (K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$, where for any non-atomic player $t \in T_1 := [0, 1]$, the action space is given by $K_t = \widehat{K} := \left\{(x_1, \ldots, x_{\overline{p}}) \in \mathbb{Z}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} x_p \leq 1\right\}$. Strategies of other players do not affect non-atomic players' admissible actions, i.e., $\Gamma_t \equiv \widehat{K}, \forall t \in T_1$. Thus, any non-atomic player can vote for a party $p \in P$ by choosing $x \in \widehat{K}$ such that $x_p = 1$, or she can abstain from voting by choosing $(x_1, \ldots, x_{\overline{p}}) = 0$.

Each $t \in T_1$ gives an importance $v_t(p) \geq 0$ to party $p \in P$ and has a favorite party $p_t^* \in P$, i.e., $v_t(p_t^*) > v_t(p)$ for all $p \in P \setminus \{p_t^*\}$. Her objective function is given by the weighted average of the utilities obtained from individual parties and a component that reflects the private level of satisfaction associated with her action, that is, for any $x = (x_1, \ldots, x_{\overline{p}}) \in K_t$,

$$u_t^\alpha(x, a) = \sum_{p=1}^{\overline{p}} v_t(p) a_p + \alpha \sum_{p=1}^{\overline{p}} (v_t(p) - \eta_t) x_p,$$

where a_p is the probability that party p wins the election, and $\eta_t \geq 0$ measures the electoral engagement of player t. Indeed, when $\alpha > 0$, as greater η_t less interested in the election would be player t. We assume that for any $t \in T_1$ either $\eta_t > v_t(p_t^*)$ or $\eta_t < v_t(p_t^*)$. The set of politically engaged players is defined as $T_1^* = \{t \in T_1 : \eta_t < v_t(p_t^*)\}$, and we assume that it is a positive measure subset of T_1.

On the other hand, there is an atomic player $T_2 = \{e\}$ whose purpose is to determine the probabilities $(a_1, \ldots, a_{\overline{p}})$ that parties have to win. These probabilities are taken as given by non-atomic players. Hence, $\Gamma_e \equiv K_e = \widehat{K}_e := \left\{(z_1, \ldots, z_{\overline{p}}) \in \mathbb{R}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} z_p = 1\right\}$ and

$$u_e(m, a) = -\sum_{p=1}^{\overline{p}} \left(a_p \sum_{p'=1}^{\overline{p}} m_{p'} - m_p \right)^2,$$

where $m = (m_1, \ldots, m_{\overline{p}})$ is the message obtained from non-atomic players' votes, assuming that $H(t, x) = x$. In other words, when a positive measure of players votes, probabilities are given by the proportion of issued votes that each party receives.

In any generalized game $\mathcal{E}_\alpha$, with $\alpha \geq 0$, the strategy chosen by a non-atomic player does not affect the social choice. However, when $\alpha > 0$, each non-atomic player gives a private value to actions and, therefore, her vote affects her utility level.

Consider the case where non-atomic players do not give importance to their strategies, i.e., $\alpha = 0$. Then, given a measurable strategy profile $x : T_1 \to \widehat{K}$ and a strategy $a \in \widehat{K}_e$, the vector

$$\begin{cases} \left(x, \left(\frac{\int_{T_1} x_p(t)dt}{\sum_{s=1}^{\overline{p}} \int_{T_1} x_s(t)dt}\right)_{p\in P}\right), & \text{if } \int_{T_1} x(t)dt \neq 0; \\ (x, a), & \text{if } \int_{T_1} x(t)dt = 0; \end{cases}$$

constitutes a Cournot-Nash equilibrium for $\mathcal{E}_0$. Therefore, when electors do not value electoral participation, there is a continuum of equilibria.

On the other hand, for any $\alpha > 0$, the generalized game $\mathcal{E}_\alpha$ has only one Cournot-Nash equilibrium. Indeed, any player $t \in T_1^*$ votes for his favorite party, while any player in $T_1 \setminus T_1^*$ does not vote. As T_1^* has positive measure, the equilibrium vector of probabilities is well defined. Hence, it follows from Theorem 1 that $\mathcal{E}_\alpha$ is an essential generalized game for any $\alpha > 0$.

Since the space $([0, 1], |\cdot|)$ is complete and $\kappa : [0, 1] \to \mathbb{G}$ given by $\kappa(\alpha) = \mathcal{E}_\alpha$ is continuous, $\mathcal{T} = (([0, 1], |\cdot|), \kappa)$ is a parametrization of $\mathbb{G}$, in the sense of Definition 5. Therefore, we conclude that $\mathcal{E}_0$—the electoral game where players do not give any value to their private strategies—has a unique $\mathcal{T}$-essential Cournot-Nash equilibrium, the one in which only politically engaged players vote in support of their favorite party. In this way, we obtain a rationale for electoral participation of politically engaged agents using essential stability as a refinement concept of Cournot-Nash equilibria.

Note that, under alternative perturbations, we can still ensure that the only essential equilibrium is that in which only politically engaged players vote. It is sufficient that only non-atomic players' payoff functions suffer perturbations, and the importance level that players give to the result of the election be small enough to maintain the same preferences over alternatives.[18]

7 Essential equilibria of discontinuous large generalized games

In this section, we extend the previous results of essential stability to a complete metric space that includes discontinuous large generalized games. Remember that, when payoff functions are discontinuous and perturbations on strategy sets are allowed, the space of large generalized games with a non-empty set of Cournot-Nash equilibria may be incomplete (see Example 3 above). For this reason, we only allow perturbations on players' objective functions.

The following concept is required to state the main assumption that ensures equilibrium existence when games are discontinuous.

Definition 9 Given a large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ and an open set $U \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$, $(\varphi_t)_{t\in T_1\cup T_2}$ are selectors of strategies supported on U when, for every $t \in T_1 \cup T_2$, $\varphi_t : U \twoheadrightarrow K_t$ is a closed correspondence with non-empty values, and for each $(m, a) \in U$, the following properties hold:

18 Perturbations on actions sets for non-atomic players, or on any atomic player characteristic, may change the underlying institutional structure, destroying the electoral dimension of the game. However, a natural perturbation in action sets is to forbid the voluntary vote, by changing K_t to $\left\{(x_1, \ldots, x_{\overline{p}}) \in \mathbb{Z}_+^{\overline{p}} : \sum_{p=1}^{\overline{p}} x_p = 1\right\}$. In this case, in any Cournot-Nash equilibrium for $\mathcal{E}_0$, all voters participate in the election. In addition, the $\mathcal{T}$-essential Cournot-Nash equilibria of $\mathcal{E}_0$ are those in which politically engaged players support their favorite party.

(i) For each $(t,k) \in T_1 \times T_2$, $\varphi_t(m,a) \times \varphi_k(m,a) \subseteq \Gamma_t(m,a) \times \Gamma_k(m,a_{-k})$.
(ii) The correspondence $t \in T_1 \twoheadrightarrow \varphi_t(m,a)$ is measurable.
(iii) For any $t \in T_2$, $\varphi_t(m,a)$ is convex.

In order to make a more clear exposition, we remind the notion of *continuous security* of a large generalized game. This concept is introduced by Barelli and Meneghel (2012) for finite-player games and generalized by Carmona and Podczeck (2014) to ensure equilibrium existence in discontinuous large generalized games.

Definition 10 (*Continuous Security*) A large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ satisfies continuous security if for every $(m,a) \notin \Lambda(\mathcal{G})$, there is an open neighborhood U of (m,a) such that, for some selectors of strategies $(\varphi_t)_{t \in T_1 \cup T_2}$ supported on U, and for some measurable function $\alpha : T_1 \cup T_2 \to [-\infty, +\infty]$, we have that

(i) For every $(m', a') \in U$, there exists a full measure set $T_1' \subseteq T_1$ satisfying

$$u_t(x, m', a') \geq \alpha(t), \quad \forall t \in T_1', \ \forall x \in \varphi_t(m', a'),$$
$$u_t(m', x, a'_{-t}) \geq \alpha(t), \quad \forall t \in T_2, \ \forall x \in \varphi_t(m', a').$$

(ii) Fix $(f', a') \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ such that $(m(f'), a') \in U$, $f'(t) \in \Gamma_t(m(f'), a')$ for almost all $t \in T_1$, and $a'_t \in \Gamma_t(m(f'), a'_{-t})$ for all $t \in T_2$. Then, either there is a positive measure set $T_1' \subseteq T_1$ such that $u_t(f'(t), m(f'), a') < \alpha(t)$, $\forall t \in T_1'$, or there is $t \in T_2$ such that $u_t(m(f'), a'_t, a'_{-t}) < \alpha(t)$.

As is shown by Carmona and Podczeck (2014), continuous security is weaker than Assumptions (A1–A2), and therefore, it is satisfied by any large generalized game in $\mathbb{G}$. Furthermore, any large generalized game satisfying continuous security has a pure strategy Nash equilibrium [see Carmona and Podczeck (2014, Theorem 1)].

To ensure that the set of discontinuous large generalized games is a complete metric space, we strengthen continuous security. With this purpose, we use the concept of *generalized payoff security* introduced by Barelli and Soza (2009) for finite-player games and extended by Carmona and Podczeck (2014) to large generalized games.

Definition 11 (*Generalized Payoff Security*) A large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ satisfies generalized payoff security if for every $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ and $\epsilon > 0$, there exists an open neighborhood U of (m,a) such that, for some selectors of strategies $(\varphi_t)_{t \in T_1 \cup T_2}$ supported on U, and for some measurable function $\alpha : T_1 \cup T_2 \to [-\infty, +\infty]$, we have that

(i) For every $(m', a') \in U$, there exists a full measure set $T_1' \subseteq T_1$ satisfying

$$u_t(x, m', a') \geq \alpha(t), \quad \forall t \in T_1', \ \forall x \in \varphi_t(m', a'),$$
$$u_t(m', x, a'_{-t}) \geq \alpha(t), \quad \forall t \in T_2, \ \forall x \in \varphi_t(m', a').$$

(ii) For any player $t \in T_2$, we have that $\alpha(t) + \epsilon \geq \sup_{x \in \Gamma_t(m, a_{-t})} u_t(m, x, a_{-t})$. In addition, the set $\{t \in T_1 : \alpha(t) + \epsilon \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a)\}$ has a measure greater than or equal to $\mu(T_1) - \epsilon$.

Definition 12 (*Upper Semicontinuous Games*) A large generalized game $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ is upper semicontinuous when the following conditions hold: (i) for each $t \in T_1$, u_t is upper semicontinuous; (ii) $\sum_{t\in T_2} u_t$ is upper semicontinuous; and (iii) for any $t \in T_1 \cup T_2$, Γ_t is upper hemicontinuous.[19]

Any $\mathcal{G}((K_t, \Gamma_t, u_t)_{t\in T_1\cup T_2})$ that is generalized payoff secure and upper semicontinuous satisfies continuous security, and therefore, it has a non-empty set of Cournot-Nash equilibria (see Lemma 2 in the Appendix). However, as the following example points out, allowing perturbations on action sets or on correspondences of admissible strategies, the collection of generalized payoff secure and upper semicontinuous games is not necessarily a ρ-complete metric space.

Example 5 Suppose that $T_1 = [0, 1]$, $T_2 = \emptyset$, $\widehat{K} = [0, 1]$ and $H(t, x) = x$. Thus, $\widehat{M} = [0, 1]$. For any $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a game with only non-atomic players, characterized by $K_t^n = \left[0, 1 - \frac{1}{n}\right]$, $\Gamma_t^n(m) = \left[0, \min\{m, 1 - \frac{1}{n}\}\right]$, and $u_t^n(f(t), m) = v(f(t))$, where $v : [0, 1] \to \{0, 1\}$ is such that $v(x) = 1$ if and only if $x = 1$. Hence, $\mathcal{G}_n$ is generalized payoff secure and upper semicontinuous.[20]

Let $\overline{\mathcal{G}}$ be the large generalized game characterized by $K_t = [0, 1]$, $\Gamma_t(m) = [0, \min\{m, 1\}]$, and $u_t(f(t), m) = v(f(t))$. It follows that $\rho(\mathcal{G}_n, \overline{\mathcal{G}})$ converges to zero as n goes to infinity. However, although $\overline{\mathcal{G}}$ is upper semicontinuous, it is not generalized payoff secure.[21] □

Taking as given $(T_1, T_2, \widehat{K}, (\widehat{K}_t)_{t\in T_2}, H, (K_t, \Gamma_t)_{t\in T_1\cup T_2})$, let $\mathbb{G}_d$ be the set of large generalized games $\mathcal{G}((u_t)_{t\in T_1\cup T_2})$ where, instead of Assumptions (A1) and (A2), *generalized payoff security* and *upper semicontinuity* hold. Recall that, in $\mathbb{G}_d$, it still requirs atomic players to have quasi-concave objective functions and convex sets of admissible strategies.

Proposition 2 *The space of discontinuous games* $(\mathbb{G}_d, \rho)$ *is complete.*

The proof is given in the Appendix.

We can adapt our previous arguments and the results of Carbonell-Nicolau (2010) to ensure that the following properties of essential stability hold.

Theorem 5 *Given a parametrization* $\mathcal{T} = ((\mathbb{X}, \tau), \kappa)$ *of* $\mathbb{G}_d$, *the collection of parameters* $\mathcal{X} \in \mathbb{X}$ *for which* $\kappa(\mathcal{X})$ *is* $\mathcal{T}$*-essential is a dense residual subset of* $\mathbb{X}$.

Furthermore, for any $\mathcal{X} \in \mathbb{X}$, *we have that*

(i) *If* $\Lambda(\kappa(\mathcal{X}))$ *is a singleton, then* $\kappa(\mathcal{X})$ *is* $\mathcal{T}$*-essential.*
(ii) *There is a minimal* $\mathcal{T}$*-essential subset of* $\Lambda(\kappa(\mathcal{X}))$.
(iii) *Every* $\mathcal{T}$*-essential and connected set* $m(\mathcal{X}) \subseteq \Lambda(\kappa(\mathcal{X}))$ *is contained in a* $\mathcal{T}$*-essential component.*

19 Given a topological space X, $u : X \to \mathbb{R}$ is upper semicontinuous if $\{x \in X : u(x) \geq a\}$ is closed for any $a \in \mathbb{R}$.

20 For every $m \in \widehat{M}$ and $\epsilon > 0$, generalized payoff security holds by choosing $\alpha \equiv 0$.

21 Taking $m = 1$ and $\epsilon \in (0, 1)$, if generalized payoff security holds for $\overline{\mathcal{G}}$, then Definition 11(i) implies that $\alpha(t) \leq 0$, $\forall t \in T_1$. On the other hand, Definition 11(ii) ensures that there exists a positive measure set $T' \subseteq T_1$ such that $\alpha(t) + \epsilon \geq \sup_{x\in\Gamma_t(1)} u_t(x, 1)$, which in turn implies that $\epsilon \geq 1$. A contradiction.

(iv) *Every $\mathcal{T}$-essential subset of $\Lambda(\kappa(\mathcal{X}))$ is stable.*

The proof is given in the Appendix.

Suppose that non-atomic players' strategies have no effect on atomic players' decisions. Then, equilibrium strategies of atomic players are a Cournot-Nash equilibrium for the finite-player game in which they are the only participants. In this context, our model captures finite-player convex games as a particular case and Theorem 5 regains previous results of Yu (1999, Theorems 4.2 and 4.3) and Carbonell-Nicolau (2010, Theorem 2).[22] Notwithstanding, we extend these previous results to generalized games, including general types of perturbations on objective functions, and adding results of existence and stability for essential sets.[23]

8 Stability of competitive prices in atomless economies

We apply the results of the previous section to a large market in which we ensure that the set of competitive prices is generically stable to perturbations on preferences.

We consider a pure exchange economy with a continuum of traders. There are L perfectly divisible commodities and a non-empty and compact metric space of consumers T_1. There is a finite measure μ and a σ-algebra $\mathcal{A}$ such that $(T_1, \mathcal{A}, \mu)$ is a complete atomless measure space.

Each $t \in T_1$ is characterized by a non-empty and compact consumption space $\widehat{K} = [0, M]^L$, a continuous and strictly increasing utility function $u_t : \widehat{K} \to \mathbb{R}_+$, and initial endowments $w(t) \in \text{int}(\widehat{K})$.

We assume that there is a finite number of agent types, i.e., there is a finite partition $\{T_{1,k}\}_{1\leq k\leq r}$ of T_1 such that, for each $k \in \{1, \ldots, r\}$ and $t, s \in T_{1,k}$ we have $(u_t, w(t)) = (u_s, w(s))$.

A *competitive equilibrium* is given by a vector of prices $\overline{p} \in \Delta := \{z \in \mathbb{R}^L_+ : \|z\|_\Sigma = 1\}$ and a consumption profile $\overline{x} : T_1 \to \widehat{K}$ such that

(i) For almost all $t \in T_1$,

$$\overline{x}(t) \in \underset{x(t)\in B_t(\overline{p})}{\text{argmax}} \ u_t(x(t)),$$

where $B_t(\overline{p}) := \{x \in \widehat{K} : \overline{p} x \leq \overline{p} w(t)\}$.

(ii) Physical markets' clearing condition hold, i.e.,

$$\int_{T_1} (\overline{x}(t) - w(t)) d\mu = 0.$$

[22] Yu (1999) also allows for perturbations on action sets and on correspondences of admissible strategies, but only for continuous games. Thus, we recover his results in Theorem 1.

[23] Scalzo (2013) extends the stability results of Carbonell-Nicolau (2010) to a space of finite-player discontinuous games where an aggregator of payoff functions satisfies a property called *generalized positively quasi-transfer continuity*. This property relaxes both generalized payoff security and upper semicontinuity. Although we focus on large generalized games where atomic players satisfy the assumptions imposed by Carbonell-Nicolau (2010), we presume that the same arguments of Scalzo (2013) may be applied to relax these assumptions in our context.

Our aim is to analyze the stability of equilibrium prices to perturbations on utility functions. Thus, leaving consumption sets and endowments fixed, let $\mathcal{E}((u_t)_{t\in T_1})$ be the economy described above. Following analogous ideas to Reny (1999, Example 3.2), define a large game $\mathcal{G}((u_t)_{t\in T_1})$ where $H(t,x) = x$ and each non-atomic player $t \in T_1$ has an strategy set $K_t = \widehat{K}$ and a payoff function $v_t : \widehat{K} \times \Delta \to \mathbb{R}$ given by

$$v_t(x,p) = \begin{cases} u_t(x), & \text{when } px \leq pw(t); \\ -1, & \text{otherwise;} \end{cases}$$

where $p \in \Delta$ is the strategy of an atomic player, denoted by a, whose objective function

$$V_a(p,m) = p\left(m - \int_{T_1} w(t)d\mu\right)$$

depends on the message $m = \int_{T_1} x(t)d\mu$ generated by non-atomic players' strategies.

Note that, the atomic player's objective function is continuous, while non-atomic players have upper semicontinuous and generalized payoff secure objective functions.[24] Thus, for every $(u_t)_{t\in T_1}$ satisfying the assumptions described above, $\mathcal{G}((u_t)_{t\in T_1}) \in \mathbb{G}_d$. Furthermore, $(m,p) \in \Lambda(\mathcal{G}((u_t)_{t\in T_1}))$ if and only if p is an equilibrium price for the economy $\mathcal{E}((u_t)_{t\in T_1})$.[25]

Let $\mathcal{T} := ((\mathbb{X},\tau),\kappa)$ be a parametrization of $\mathbb{G}_d$ such that, for every $\mathcal{X} \in \mathbb{X}$, there are continuous and strictly increasing utility functions $(u_t^{\mathcal{X}})_{t\in T_1}$ such that $\kappa(\mathcal{X}) = \mathcal{G}((u_t^{\mathcal{X}})_{t\in T_1})$. It follows that a Cournot-Nash equilibrium of $\mathcal{G}((u_t^{\mathcal{X}})_{t\in T_1})$ is $\mathcal{T}$-essential if and only if there exists an equilibrium price of $\mathcal{E}((u_t^{\mathcal{X}})_{t\in T_1})$ that is stable to the perturbations on utility functions determined by $\mathcal{T}$.

24 Utility functions $(u_t)_{t\in T_1}$ are continuous and take nonnegative values. Since for every $t \in T_1$ the initial endowment $w(t) \in \text{int}(\widehat{K})$, the budget set correspondence $p \in \Delta \twoheadrightarrow B_t(p)$ is continuous and has non-empty and compact values. Therefore, Berge's Maximum Theorem [see Aliprantis and Border (2006, Theorem 17.31, page 570)] guarantees that, given $(m,p) \in \widehat{K} \times \Delta$ and $\epsilon > 0$, generalized payoff security holds by choosing a sufficient small neighborhood U of (m,p) and mappings $\alpha : T_1 \cup \{a\} \to \mathbb{R}$ and $(\varphi_t)_{t\in T_1\cup\{a\}}$ such that $\alpha(a) = V_a(\varphi_0(m,p),m) - \epsilon, \varphi_a(m',p') = \text{argmax}_{p''\in\Delta} V_a(p'',m'), \forall (m',p') \in U$, and for each non-atomic player $t \in T_1, \alpha(t) = u_t(\varphi_t(m,p)) - \epsilon$ and $\varphi_t(m',p') = \text{argmax}_{x(t)\in B_t(p')} u_t(x(t)), \forall (m',p') \in U$. The existence of finitely many types of non-atomic agents guarantees that there is a common neighborhood U for every $t \in T_1 \cup \{a\}$ and also ensures that for every $(m',p') \in U$, the map $t \in T_1 \to (\alpha(t), \varphi_t(m',p'))$ is measurable.

25 Since $\mathcal{G}((u_t)_{t\in T_1}) \in \mathbb{G}_d$, it has a non-empty set of Cournot-Nash equilibria. As functions $(u_t)_{t\in T_1}$ take nonnegative values and are strictly increasing, $(\overline{m},\overline{p}) \in \Lambda(\mathcal{G}((u_t)_{t\in T_1}))$ if and only if for some $\overline{x} \in \widehat{\mathcal{F}}^1$ we have that

(i) There exists a full measure set $T_1' \subseteq T_1$ such that $v_t(\overline{x}(t),\overline{p}) = u_t(\overline{x}(t)) = \max_{x(t)\in B_t(\overline{p})} u_t(x(t)), \forall t \in T_1'$.
(ii) $\overline{p} \gg 0$ and $\overline{p}(\overline{x}(t) - w(t)) = 0, \forall t \in T_1'$.
(iii) $\overline{m} = \int_{T_1} \overline{x}(t)d\mu = \int_{T_1} w(t)d\mu$.

Thus, $(\overline{m},\overline{p}) \in \Lambda(\mathcal{G}((u_t)_{t\in T_1}))$ if and only if there exists $\overline{x} \in \widehat{\mathcal{F}}^1$ such that $(\overline{p},\overline{x})$ is an equilibrium for $\mathcal{E}((u_t)_{t\in T_1})$.

It follows from Theorem 5 that the set of parameters $\mathcal{X} \in \mathbb{X}$ for which all competitive equilibrium prices of $\mathcal{E}((u_t^{\mathcal{X}})_{t\in T_1})$ are stable to perturbations generated by $\mathcal{T}$ is a dense residual subset of $\mathbb{X}$. Therefore, competitive equilibria are generically stable when perturbations are determined by continuous parametrizations that preserve both the continuity and the strict monotonicity of individuals' preferences.

9 Concluding remarks

In this paper, we use the stability theory of fixed points developed by Fort (1950) and Jiang (1962) to address the essential stability of Cournot-Nash equilibria in large generalized games.

We guarantee that essential stability is a generic property in the space of continuous large generalized games. Essential equilibria are still generic when large generalized games are generalized payoff secure and upper semicontinuous, provided that only payoff perturbations be allowed. Also, all games have essential subsets of the set of equilibria, which varies continuously.

Our results are compatible with general types of perturbations on the characteristics of generalized games. Indeed, stability properties still hold when (i) admissible perturbations can be captured by a continuous parametrization of the set of generalized games; and (ii) the set of parameters constitutes a complete metric space.

10 Appendix

Lemma 1 *The set of messages $\widehat{M}$ is non-empty and compact.*

Proof Since $t \in T_1 \twoheadrightarrow K_t$ is measurable, from Aliprantis and Border (2006, Lemma 18.2, and Theorem 18.6), we know that this correspondence has an $\mathcal{A} \times \mathcal{B}(\widehat{K})$-measurable graph. It follows from Aumann's Selection Theorem [see Aliprantis and Border (2006, Theorem 18.26, page 608)] that there exists an $\mathcal{A}$-measurable function $g : T_1 \to \widehat{K}$ such that, $g(t) \in K_t$, $\forall t \in T_1$. Hence, the compactness of T_1 and $\widehat{K}$ and the continuity of H guarantee that the map $t \to H(t, g(t))$ is bounded and measurable. Therefore, $\widehat{M}$ is non-empty.

Since H is continuous, μ is a finite measure, and the sets T_1 and $\widehat{K}$ are compact, the correspondence $t \in T_1 \twoheadrightarrow H(t, \widehat{K})$ is integrable bounded and has closed values. Thus, as $\widehat{M} = \int_{T_1} H(t, \widehat{K}) d\mu$, it follows from Aumann (1965, Theorem 4) that $\widehat{M}$ is compact. □

Let us define some notations. Given a metric space (S, d), consider the sets

$$A(S) = \{K \subseteq S : K \text{ is non-empty and compact}\},$$
$$A_c(S) = \{C \subseteq A(S) : C \text{ is convex}\}.$$

Denote by d_H the Hausdorff metric induced by the metric of S. If S is compact, then $(A(S), d_H)$ is a complete metric space. Also, when S is compact and convex, $(A_c(S), d_H)$ is complete.[26]

Given a set X, let $\mathcal{U}(X)$ be the collection of bounded functions $u : X \to \mathbb{R}$ endowed with the sup norm topology, i.e., the topology determined by the metric $d(u_1, u_2) = \sup_{x\in X} |u_1(x) - u_2(x)|$.

Proof of Proposition 1 Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, with $\mathcal{G}_n = \mathcal{G}_n((K_{n,t}, \Gamma_{n,t}, u_{n,t})_{t\in T_1\cup T_2})$, be a Cauchy sequence on $(\mathbb{G}, \rho)$. It follows from definition of $\mathbb{G}$ and ρ that for any non-atomic player $t \in T_1$, $\{K_{n,t}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A(\widehat{K}), d_H)$. Also, for any atomic player $s \in T_2$, $\{K_{n,s}\}_{n\in\mathbb{N}}$ is a Cauchy sequence on $(A_c(\widehat{K}_s), d_{H,s})$. Hence, there are sets $\{\overline{K}_t\}_{t\in T_1\cup T_2}$ such that (i) $(\overline{K}_t, \overline{K}_s) \in A(\widehat{K}) \times A_c(\widehat{K}_s)$, $\forall (t, s) \in T_1 \times T_2$; and (ii) for any $(t, s) \in T_1 \times T_2$, we have that $\lim_{n\to+\infty} d_H(K_{n,t}, \overline{K}_t) = \lim_{n\to+\infty} d_{H,s}(K_{n,s}, \overline{K}_s) = 0$.

The definition of the metric ρ ensures that, for any $t \in T_1$ and $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the sequence $\{\Gamma_{n,t}(m, a)\}_{n\in\mathbb{N}} \subseteq A(\widehat{K})$ is Cauchy and, therefore, there exists a set $K_t(m, a) \in A(\widehat{K})$ such that $d_H(\Gamma_{n,t}(m, a), K_t(m, a))$ converges to zero as n goes to infinity. Let $\overline{\Gamma}_t : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{K}$ be the set-valued mapping defined by $\overline{\Gamma}_t(m, a) = K_t(m, a)$. It follows that correspondences $(\overline{\Gamma}_t)_{t\in T}$ are continuous.[27] By analogous arguments, we can obtain that for any $s \in T_2$, there is a continuous correspondence $\overline{\Gamma}_s : \widehat{M} \times \widehat{\mathcal{F}}^2_{-s} \twoheadrightarrow \widehat{K}_s$ such that, for each $(m, a_{-s}) \in \widehat{M} \times \widehat{\mathcal{F}}^2_{-s}$ both $\overline{\Gamma}_s(m, a_{-s}) \in A_c(\widehat{K}_s)$ and $d_{H,s}(\Gamma_{n,s}(m, a_{-s}), \overline{\Gamma}_s(m, a_{-s}))$ converges to zero as n increases.

Since $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is Cauchy on $(\mathbb{G}, \rho)$, there is a bounded function $U : T_1 \times \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{R}$ such that, for each $t \in T_1$, the sequence $\{u_{n,t}\}_{n\in\mathbb{N}} \subseteq \mathcal{U}(\widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2)$ converges uniformly to $\overline{u}_t := U(t, \cdot)$ and, therefore, $\overline{u}_t$ is continuous. Analogously, for any $t \in T_2$, the sequence $\{u_{n,t}\}_{n\in\mathbb{N}} \subseteq \mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ converges to some continuous function $\overline{u}_t \in \mathcal{U}(\widehat{M} \times \widehat{\mathcal{F}}^2)$ that is quasi-concave on a_t.

Let $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t, \overline{\Gamma}_t, \overline{u}_t)_{t\in T_1\cup T_2})$. It follows from arguments above that $\lim_{n\to+\infty} \rho(\mathcal{G}_n, \overline{\mathcal{G}}) = 0$. Thus, to conclude that $(\mathbb{G}, \rho)$ is complete, it is sufficient to guarantee that

(i) for each $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $(t, x) \in T_1 \times \widehat{K} \to \overline{u}_t(x, m, a)$ is measurable;
(ii) for each $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m, a)$ is measurable;
(iii) the correspondence $t \in T_1 \twoheadrightarrow \overline{K}_t$ is measurable.

Fix $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, the definition of ρ ensures that measurable functions $(t, x) \in T_1 \times \widehat{K} \to u_{n,t}(x, m, a)$ converge to the mapping $(t, x) \in T_1 \times \widehat{K} \to \overline{u}_t(x, m, a)$.

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26 Since (S, d) is a compact metric space, it follows from Aliprantis and Border (2006, Theorem 3.85-(3) and Theorem 3.88-(2), pages 116 and 119, respectively) that $A(S)$ is a complete metric space under the Hausdorff distance induced by d. When S is a compact subset of a normed vector space, $(A_c(S), d_H)$ remains a complete metric space, since the Hausdorff limit of a sequence of compact convex sets is still a compact convex set.

27 Since $\widehat{M} \times \widehat{\mathcal{F}}^2$ is compact and $(A(\widehat{K}), d_H)$ is complete, for every $t \in T_1$, the continuity of the correspondence $\overline{\Gamma}_t$ follows from the completeness of the space of continuous functions $\nu : \widehat{M} \times \widehat{\mathcal{F}}^2 \to A(\widehat{K})$ under the uniform metric induced by the Haussdorf distance. Indeed, any correspondence $\Gamma : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{K}$ with non-empty and compact values can be identified with the function $B_\Gamma : \widehat{M} \times \widehat{\mathcal{F}}^2 \to A(\widehat{K})$ given by $B_\Gamma(m, a) = \Gamma(m, a)$, in such form that Γ is continuous if and only if B_Γ is continuous [see Aliprantis and Border (2006, Lemma 3.97 and Theorem 17.15, pages 124 and 563)].

Since $(T_1 \times \widehat{K}, \mathcal{A} \times \mathcal{B}(\widehat{K}))$ is a measurable space, item (i) holds [see Aliprantis and Border (2006, Lemma 4.29, page 142)]. Furthermore, since for every $n \in \mathbb{N}$ the correspondence $t \in T_1 \twoheadrightarrow \Gamma_{n,t}(m,a)$ is measurable, it follows from Aliprantis and Border (2006, Theorem 18.10, page 598) that the function $\Theta_{n,(m,a)} : T_1 \to A(\widehat{K})$ defined by $\Theta_{n,(m,a)}(t) = \Gamma_{n,t}(m,a)$ is Borel measurable. Also, the sequence $\{\Theta_{n,(m,a)}\}_{n\in\mathbb{N}}$ converges to $\overline{\Theta}_{(m,a)} : T_1 \to A(\widehat{K})$, where $\overline{\Theta}_{(m,a)}(t) = \overline{\Gamma}_t(m,a)$. By Aliprantis and Border (2006, Lemma 4.29), $\overline{\Theta}_{(m,a)}$ is a Borel measurable function. Thus, $t \in T_1 \twoheadrightarrow \overline{\Gamma}_t(m,a)$ is measurable [see Aliprantis and Border (2006, Theorem 18.10)]. By analogous arguments, we obtain item (iii). □

Proof of Theorem 1 The proof is a direct consequence of the following steps.

Step 1. $\Lambda : \mathbb{G} \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ is upper hemicontinuous with compact values.
Since $\widehat{M} \times \widehat{\mathcal{F}}^2$ is compact and non-empty, we only need to prove that Graph(Λ) is closed, where Graph$(\Lambda) := \left\{(\mathcal{G},(m,a)) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2 : (m,a) \in \Lambda(\mathcal{G})\right\}$. Let $\{(\mathcal{G}_n,(m_n,a_n))\}_{n\in\mathbb{N}} \subset$ Graph(Λ) be a sequence converging to $(\overline{\mathcal{G}},(\overline{m},\overline{a})) \in \mathbb{G} \times \widehat{M} \times \widehat{\mathcal{F}}^2$, where $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{K}_t,\overline{\Gamma}_t,\overline{u}_t)_{t\in T_1\cup T_2})$ and, for every $n \in \mathbb{N}, \mathcal{G}_n = \mathcal{G}_n((K_t^n,\Gamma_t^n,u_t^n)_{t\in T_1\cup T_2})$.

We aim to ensure that $(\overline{m},\overline{a}) \in \Lambda(\overline{\mathcal{G}})$. Since for any $n \in \mathbb{N}$, $(m_n,a_n) \in \Lambda(\mathcal{G}_n)$, there exists $f_n \in \widehat{\mathcal{F}}^1$ such that (i) the function $g_n : T_1 \to \mathbb{R}^m$ given by $g_n(t) = H(t, f_n(t))$ is measurable and $m_n = m(f_n)$; and (ii) for almost all $t \in T_1$ both $f_n(t) \in \Gamma_t^n(m_n,a_n)$ and

$$u_t^n(f_n(t),m_n,a_n) = \max_{x\in\Gamma_t^n(m_n,a_n)} u_t^n(x,m_n,a_n).$$

Claim A *There exists $\overline{f} \in \widehat{\mathcal{F}}^1$ such that $\overline{m} = \int_{T_1} H(t,\overline{f}(t))d\mu$.*

Proof Since H is continuous, T_1 is compact and $\{f_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{F}}^1$, it follows that the sequence $\{g_n\}_{n\in\mathbb{N}}$ is uniformly integrable [see Hildenbrand (1974, page 52)]. In addition, $\{\int_{T_1} g_n(t)d\mu\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$ converges to $\overline{m}$ as n goes to infinity , and therefore, the multidimensional version of Fatou's Lemma [see Hildenbrand (1974, page 69)] guarantees that there is $g : T_1 \to \mathbb{R}^m$ integrable such that,[28]

(1) $\overline{m} = \lim_{n\to\infty} \int_{T_1} g_n(t)d\mu = \int_{T_1} g(t)d\mu$;

(2) there exists a full measure set $\tilde{T}_1 \subseteq T_1$ such that, for any $t \in \tilde{T}_1$, $g(t) \in Ls(g_n(t))$, where $Ls(g_n(t))$ is the set of cluster points of $\{g_n(t)\}_{n\in\mathbb{N}}$.[29]

Fix $t \in \tilde{T}_1$. Then, there is a subsequence $(g_{n_k}(t))_k$ converging to $g(t)$. Since $\{f_{n_k}(t)\}_{k\in\mathbb{N}} \subseteq \widehat{K}$, by taking a subsequence if it is necessary, we can ensure that there exists $f(t) \in \widehat{K}$ such that both $f_{n_k}(t) \to f(t)$ and $g(t) = \lim_{k\to\infty} H(t, f_{n_k}(t)) = H(t, f(t))$ hold.

[28] Although maps $\{g_n\}_{n\in\mathbb{N}}$ can take negative values, they are uniformly bounded from below (since $\widehat{K}$ and T_1 are compact sets, and H is continuous). Thus, as T_1 has finite Lebesgue measure, we can apply the Fatou's Lemma.

[29] In other words, for any $t \in \tilde{T}_1$, there is at least one subsequence of $\{g_n(t)\}_{n\in\mathbb{N}}$ converging to $g(t)$.

Let $\overline{f} : T_1 \to \widehat{K}$ be a function such that

$$\overline{f}(t) \in \begin{cases} \{f(t)\}, & \text{if } t \in \tilde{T}_1, \\ \overline{\Gamma}_t(\overline{m}, \overline{a}), & \text{if } t \in T \setminus \tilde{T}_1. \end{cases}$$

Then, it follows that

$$\overline{m} = \lim_{n\to\infty} m_n = \lim_{n\to\infty} \int_{T_1} g_n(t) d\mu = \int_{T_1} g(t) d\mu = \int_{T_1} H(t, \overline{f}(t)) d\mu,$$

where the last equality follows from the fact that $T_1 \setminus \tilde{T}_1$ has zero measure. □

Claim B *For almost all* $t \in T_1, \overline{f}(t) \in \overline{\Gamma}_t(\overline{m}, \overline{a})$. *In addition, for any* $t \in T_2, \overline{a}_t \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$.

Proof Following the notation of the proof of the previous claim, fix $t \in \tilde{T}_1$ and let $\{f_{n_k}(t)\}_{k\in\mathbb{N}}$ be the sequence converging to $\overline{f}(t)$ and that was obtained in the previous claim. We know that, for any $k \in \mathbb{N}$, $f_{n_k}(t) \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})$. Therefore,

$$\begin{aligned} d(\overline{f}(t), \overline{\Gamma}_t(\overline{m}, \overline{a})) &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(m_{n_k}, a_{n_k})) \\ &\quad + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \\ &\le \widehat{d}(\overline{f}(t), f_{n_k}(t)) + \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})), \end{aligned}$$

where $\widehat{d}$ denotes the metric of the compact metric space $\widehat{K}$. Since $\overline{\Gamma}_t$ is continuous, by taking the limit as k goes to infinity, we obtain the first property.

On the other hand, for any $(t, n) \in T_2 \times \mathbb{N}$, $a_{n,t} \in \Gamma_t^n(m_n, a_{n,-t})$, which implies that

$$\begin{aligned} d(\overline{a}_t, \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + d_{H,t}(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(m_n, a_{n,-t})) \\ &\quad + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \\ &\le \widehat{d}_t(\overline{a}_t, a_{n,t}) + \rho(\mathcal{G}_n, \overline{\mathcal{G}}) + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})), \end{aligned}$$

where $\widehat{d}_t$ denotes the metric of $\widehat{K}_t$. Taking the limit as n goes to infinity, we obtain the result. □

Claim C *The following properties hold:*

(i) For almost all $t \in T_1, \overline{f}(t) \in argmax_{x\in\overline{\Gamma}_t(\overline{m},\overline{a})}\ \overline{u}_t(x, \overline{m}, \overline{a})$.
(ii) For any $t \in T_2, \overline{a}_t \in argmax_{x\in\overline{\Gamma}_t(\overline{m},\overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$.

Proof (i) Given $t \in \tilde{T}_1$, we have that

$$d_H(\Gamma_t^{n_k}(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})) \le \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + d_H(\overline{\Gamma}_t(m_{n_k}, a_{n_k}), \overline{\Gamma}_t(\overline{m}, \overline{a})).$$

Then, $\Gamma_t^{n_k}(m_{n_k}, a_{n_k}) \longrightarrow_k \overline{\Gamma}_t(\overline{m}, \overline{a})$. Since $u_t^{n_k}$ converges uniformly to $\overline{u}_t$, it follows from Yu (1999, Lemma 2.5) and Aubin (1982, Theorem 3, page 70) that

$$u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) = \max_{x \in \Gamma_t^{n_k}(m_{n_k}, a_{n_k})} u_t^{n_k}(x, m_{n_k}, a_{n_k}) \longrightarrow_k \max_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a})$$

On the other hand,

$$|u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})| \\ \leq \rho(\mathcal{G}_{n_k}, \overline{\mathcal{G}}) + |\overline{u}_t(f_{n_k}(t), m_{n_k}, a_{n_k}) - \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})|.$$

Taking the limit as k goes to infinity, we obtain that $u_t^{n_k}(f_{n_k}(t), m_{n_k}, a_{n_k}) \rightarrow \overline{u}_t(\overline{f}(t), \overline{m}, \overline{a})$. Hence, it follows from Claim B that $\overline{f}(t) \in \underset{x \in \overline{\Gamma}_t(\overline{m}, \overline{a})}{\operatorname{argmax}} \overline{u}_t(x, \overline{m}, \overline{a})$.
(ii) Given $t \in T_2$, analogous arguments to those made in the previous item ensure that

$$d_{H,t}(\Gamma_t^n(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})) \leq \rho(\mathcal{G}_n, \mathcal{G}) + d_{H,t}(\overline{\Gamma}_t(m_n, a_{n,-t}), \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})),$$

which implies that $\Gamma_t^n(m_n, a_{n,-t})$ converges to $\overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})$ as n goes to infinity. Hence, Yu (1999, Lemma 2.5) ensures that

$$u_t^n(m_n, a_n) = \max_{x \in \Gamma_t^n(m_n, a_{n,-t})} u_t^n(m_n, x, a_{n,-t}) \longrightarrow \max_{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t}).$$

Since $\lim_{n \to +\infty} u_t^n(m_n, a_n) = \overline{u}_t(\overline{m}, \overline{a})$,[30] $\overline{a}_t \in \underset{x \in \overline{\Gamma}_t(\overline{m}, \overline{a}_{-t})}{\operatorname{argmax}} \overline{u}_t(\overline{m}, x, \overline{a}_{-t})$. □

It follows from Claims A and C that $(\overline{m}, \overline{a}) \in \Lambda(\overline{\mathcal{G}})$. Thus, we ensure that Λ is an upper hemicontinuous correspondence with compact values.

Step 2. There is a dense residual set $Q \subseteq \mathbb{G}'$ where Λ is lower hemicontinuous.
Since $\mathbb{G}'$ is a closed subset of $\mathbb{G}$, it follows that $(\mathbb{G}', \rho)$ is a complete metric space and, therefore, it is a Baire space. Since the correspondence Λ is non-empty, compact-valued, and upper hemicontinuous, it follows from Lemmas 5 and 6 in Carbonell-Nicolau (2010) [see also Fort (1949) and Jiang (1962)] that there exists a dense residual subset Q of $\mathbb{G}'$ in which Λ is lower hemicontinuous.

Step 3. If $\mathcal{G} \in \mathbb{G}'$ is a point of lower hemicontinuity of Λ, then $\mathcal{G}$ is essential with respect to $\mathbb{G}'$.
Fix $(f^*, a^*) \in \mathrm{CN}(\mathcal{G})$. Then, for any open neighborhood $O \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ of $(m(f^*), a^*)$, we have $\Lambda(\mathcal{G}) \cap O \neq \emptyset$, and therefore, by the lower hemicontinuity, we have that $\{\mathcal{G}' \in \mathbb{G}' : \Lambda(\mathcal{G}') \cap O \neq \emptyset\}$ contains a neighborhood of $\mathcal{G}$, that is, for some $\epsilon > 0$ and for any $\mathcal{G}' \in \mathbb{G}'$ such that $\rho(\mathcal{G}', \mathcal{G}) < \epsilon$, we have $\Lambda(\mathcal{G}') \cap O \neq \emptyset$. Hence, all Cournot-Nash equilibria of $\mathcal{G}$ are essential with respect to $\mathbb{G}'$.

[30] It is a direct consequence of the fact that, for any $n \in \mathbb{N}$, we have

$$|u_t^n(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})| \leq \rho(\mathcal{G}_n, \mathcal{G}) + |\overline{u}_t(m_n, a_n) - \overline{u}_t(\overline{m}, \overline{a})|.$$

It follows from Steps 2 and 3 that any game in Q is essential.

Finally, suppose that for some $\mathcal{G} \in \mathbb{G}'$ the set $\Lambda(\mathcal{G})$ is a singleton. Then, the upper hemi-continuity of Λ guarantees that it is continuous at $\mathcal{G}$. Finally, Step 3 implies that $\mathcal{G}$ is an essential generalized game with respect to $\mathbb{G}'$. □

Proof of Theorem 2 (i) Existence of a minimal essential set.
Fix $\mathcal{G} \in \mathbb{G}'$. Let $\mathcal{S}$ be the family of essential subsets of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$ ordered by set inclusion. Since Λ is upper hemicontinuous, $\Lambda(\mathcal{G}) \in \mathcal{S}$ and, hence, $\mathcal{S} \neq \emptyset$. As any essential set is non-empty and compact, each totally ordered subset of $\mathcal{S}$ has a lower bound. By Zorn's Lemma, $\mathcal{S}$ has a minimal element and, by definition, it is an essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$.
(ii) If there are connected essential sets, then there are essential components.
Suppose that there is a connected essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$, denoted by $c(\mathcal{G})$. Since $c(\mathcal{G})$ is non-empty, fix $(\widehat{m}, \widehat{a}) \in c(\mathcal{G})$ and consider the set $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ defined as the union of all connected subsets of $\Lambda(\mathcal{G})$ that contains $(\widehat{m}, \widehat{a})$. By definition, $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is a component of $\Lambda(\mathcal{G})$. As the closure of a connected set is connected and $\Lambda(\mathcal{G})$ is compact, it follows that $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is compact. Hence, the essentiality of $c(\mathcal{G}) \subseteq \Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ with respect to $\mathbb{G}'$ ensures that the component $\Lambda_{(\widehat{m},\widehat{a})}(\mathcal{G})$ is also an essential subset of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$.
(iii) Connectedness of minimal essential sets in normed spaces.
Suppose that $\widehat{K}$ is a convex subset of a normed space and it is equipped with a metric induced by a norm. Fix a minimal essential set of $\Lambda(\mathcal{G})$ with respect to $\mathbb{G}'$, denoted by $m(\mathcal{G})$. Suppose, by contradiction, that $m(\mathcal{G})$ is disconnected.

Claim A *There are open sets $U_1, U_2 \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ such that $m(\mathcal{G}) \subset U_1 \cup U_2$ and $\overline{U}_1 \cap \overline{U}_2 = \emptyset$.*

Proof Since $m(\mathcal{G})$ is disconnected, there are closed and non-empty subsets $A_1, A_2 \subseteq \Lambda(\mathcal{G})$ such that $A_1 \cap A_2 = \emptyset$ and $m(\mathcal{G}) = A_1 \cup A_2$. Since $m(\mathcal{G})$ is minimal, neither A_1 nor A_2 are essential with respect to $\mathbb{G}'$. Hence, for each $i \in \{1, 2\}$, there exists an open set U_i such that $A_i \subset U_i$ and for all $\epsilon > 0$, there exists $\mathcal{G}^i_\epsilon \in \mathbb{G}'$ such that both $\rho(\mathcal{G}, \mathcal{G}^i_\epsilon) < \epsilon$ and $\Lambda(\mathcal{G}^i_\epsilon) \cap U_i = \emptyset$. Since A_i is compact, we can assume that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$. □

Claim B *There are large generalized games $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{G}'$ such that*

$$\Lambda(\mathcal{G}_i) \cap U_i = \emptyset \quad \wedge \quad \Lambda(\mathcal{G}_i) \cap U_j \neq \emptyset, \qquad \forall i, j \in \{1, 2\} : i \neq j.$$

In addition, there is a continuous map $G : \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{G}$ such that, for every $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$,

$$\Lambda(G(m, a)) \cap (U_1 \cup U_2) \neq \emptyset, \qquad (G(m, a) = \mathcal{G}_i \iff (m, a) \in \overline{U}_i), \ \forall i \in \{1, 2\}.$$

Proof As $m(\mathcal{G})$ is essential with respect to $\mathbb{G}'$, there exists $\nu > 0$ such that for every $\mathcal{G}' \in \mathbb{G}'$ with $\rho(\mathcal{G}, \mathcal{G}') < \nu$, we have $\Lambda(\mathcal{G}') \cap (U_1 \cup U_2) \neq \emptyset$. Following the notation of the previous claim, for each $i \in \{1, 2\}$, set $\mathcal{G}_i = \mathcal{G}^i_{\nu/3}$. Hence, for $i \neq j$, we obtain $\Lambda(\mathcal{G}_i) \cap U_j \neq \emptyset$.

Let $G : \widehat{M} \times \widehat{\mathcal{F}}^2 \to \mathbb{G}$ be the function[31]

$$G(m,a) = \lambda(m,a)\mathcal{G}_1 + (1-\lambda(m,a))\mathcal{G}_2, \quad \forall (m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2,$$

where $\lambda : \widehat{M} \times \widehat{\mathcal{F}}^2 \to [0,1]$ is the continuous function given by,

$$\lambda(m,a) = \frac{d((m,a),\overline{U}_2)}{d((m,a),\overline{U}_1) + d((m,a),\overline{U}_2)}.$$

By construction, for each $i \in \{1,2\}$, $G(m,a) = \mathcal{G}_i$ if and only if $(m,a) \in \overline{U}_i$.

Since metric spaces $\widehat{K}$ and $\{\widehat{K}_t\}_{t\in T_2}$ are contained in normed vectorial spaces and their metrics are induced by norms, for any $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, we can ensure that $G(m,a)$ is well defined and

$$\begin{aligned}\rho(G(m,a),\mathcal{G}_1) &= \rho\left(\lambda(m,a)\mathcal{G}_1 + (1-\lambda(m,a))\mathcal{G}_2, \lambda(m,a)\mathcal{G}_1 + (1-\lambda(m,a))\mathcal{G}_1\right)\\ &= \lambda(m,a)\rho(\mathcal{G}_1,\mathcal{G}_1) + (1-\lambda(m,a))\rho(\mathcal{G}_2,\mathcal{G}_1)\\ &\le \rho(\mathcal{G}_2,\mathcal{G}_1) \le \rho(\mathcal{G}_2,\mathcal{G}) + \rho(\mathcal{G},\mathcal{G}_1) < \frac{2\nu}{3},\end{aligned}$$

which implies that $\rho(\mathcal{G}, G(m,a)) \le \rho(\mathcal{G},\mathcal{G}_1) + \rho(\mathcal{G}_1, G(m,a)) < \nu$. Hence, for each $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$, $\Lambda(G(m,a)) \cap (U_1 \cup U_2) \neq \emptyset$. □

Given a large generalized game $\mathcal{G} \in \mathbb{G}$, let $\Phi_{\mathcal{G}} : \widehat{M} \times \widehat{\mathcal{F}}^2 \twoheadrightarrow \widehat{M} \times \widehat{\mathcal{F}}^2$ be the correspondence defined by $\Phi_{\mathcal{G}}(m,a) = \left(\Omega^{\mathcal{G}}(m,a), (B_t^{\mathcal{G}}(m,a_{-t}))_{t\in T_2}\right)$, where

$$\Omega^{\mathcal{G}}(m,a) := \left\{ \int_{T_1} H(t,f(t))d\mu : H(\cdot,f(\cdot)) \text{ is integrable} \wedge f(t) \in B_t^{\mathcal{G}}(m,a),\ \forall t \in T_1 \right\};$$

$$B_t^{\mathcal{G}}(m,a) := \operatorname*{argmax}_{x_t \in \Gamma_t(m,a)} u_t(x_t,m,a), \quad \forall t \in T_1;$$

$$B_t^{\mathcal{G}}(m,a_{-t}) := \operatorname*{argmax}_{x_t \in \Gamma_t(m,a_{-t})} u_t(x_t,m,a_{-t}), \quad \forall t \in T_2.$$

We affirm that $\Phi_{\mathcal{G}}$ is upper hemicontinuous and has non-empty, compact, and convex values. Note that, Berge's Maximum Theorem [see Aliprantis and Border (2006, Theorem 17.31, page 570)] guarantees that for any $t \in T_1 \cup T_2$, the correspondence $B_t^{\mathcal{G}}$ is upper hemicontinuous and has non-empty and compact values. Also, as atomic players have convex strategy sets and quasi-concave payoff functions, correspondences $(B_t^{\mathcal{G}})_{t\in T_2}$ have convex values. Hence, we want to prove that $\Omega^{\mathcal{G}}(\cdot) = \int_{T_1} H(t, B_t^{\mathcal{G}}(\cdot))d\mu$ is upper hemicontinuous and has non-empty, compact, and convex values. Aumann (1965, Theorem 1) guarantees that $\Omega^{\mathcal{G}}$ has convex values. Fix $(m,a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$. Since $t \in T_1 \twoheadrightarrow \Gamma_t(m,a)$ is measurable, it follows

[31] The function G is well defined, because $\widehat{K}$ and $\widehat{K}_t$, where $t \in T_2$, are convex subsets of normed spaces with metrics induced by norms (see footnote 15).

from Aliprantis and Border (2006, Lemma 18.2, page 593) that $t \in T_1 \twoheadrightarrow \Gamma_t(m,a)$ is weakly measurable. The Measurable Maximum Theorem (Aliprantis and Border (2006, Theorem 18.19, page 605)) implies that $t \in T_1 \twoheadrightarrow H(t, B_t^{\mathcal{G}}(m,a))$ has a measurable selector. Since H is continuous, the compactness of T_1 and $\widehat{K}$ guarantees that $t \in T_1 \twoheadrightarrow H(t, B_t^{\mathcal{G}}(m,a))$ is bounded and, therefore, its measurable selectors are integrable. We conclude that $\Omega^{\mathcal{G}}(m,a)$ is non-empty. Since T_1 has finite measure and $t \in T_1 \twoheadrightarrow H(t, B_t^{\mathcal{G}}(m,a))$ is bounded and has closed values, it follows from Aumann (1965, Theorem 4) that $\Omega^{\mathcal{G}}(m,a)$ is compact. Finally, as H is continuous, $\widehat{K}$ is compact, and for every $t \in T_1$ the correspondence $(m,a) \twoheadrightarrow B_t^{\mathcal{G}}(m,a)$ has closed graph; it follows that $(m,a) \twoheadrightarrow H(t, B_t^{\mathcal{G}}(m,a))$ is upper hemicontinuous for each $t \in T_1$. Thus, it follows from Aumann (1965, Corollary 5.2) that $\Omega^{\mathcal{G}}$ is upper hemicontinuous.

Therefore, Kakutani's Fixed Point Theorem implies that the set of fixed points of $\Phi_{\mathcal{G}}$ is non-empty and compact. Note that (f^*, a^*) is a Cournot-Nash equilibrium of $\mathcal{G}$ if and only if $(m^*, a^*) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ is a fixed point of $\Phi_{\mathcal{G}}$, where $m^* = \int_{T_1} H(t, f^*(t))d\mu$.[32]

Claim C *There exists $(\overline{m}, \overline{a}) \in U_1$ such that $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$.*

Proof Given a compact, convex, and non-empty set $\tilde{A}_1 \subset U_1$, let $\Theta : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1 \times \tilde{A}_1$ be the correspondence defined by $\Theta((m_1,a_1),(m_2,a_2)) = \left(\Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1\right) \times \{(m_1,a_1)\}$. If the set-valued map $\Theta_1 : \tilde{A}_1 \times \tilde{A}_1 \twoheadrightarrow \tilde{A}_1$ given by $\Theta_1((m_1,a_1),(m_2,a_2)) = \Phi_{G(m_1,a_1)}(m_2,a_2) \cap \tilde{A}_1$ has closed graph, then Θ is upper hemicontinuous and has non-empty, compact, and convex values. Thus, applying Kakutani's Fixed Point Theorem, we could find $(\overline{m}, \overline{a}) \in \tilde{A}_1 \subset U_1$ such that $(\overline{m}, \overline{a}) \in \Lambda(G(\overline{m}, \overline{a}))$.

Therefore, to prove the claim, it is sufficient to ensure that Θ_1 has closed graph. Fix a sequence $\{(z_1^n, z_2^n, (m^n, a^n))\}_{n \in \mathbb{N}} \subset \mathrm{Graph}(\Theta_1)$ that converges to $(\tilde{z}_1, \tilde{z}_2, (\tilde{m}, \tilde{a}))$. We aim to guarantee that $(\tilde{m}, \tilde{a}) \in \Theta_1(\tilde{z}_1, \tilde{z}_2)$.

For convenience of notations, assume that $\mathcal{G}_i = \mathcal{G}_i((K_t^i, \Gamma_t^i, u_t^i)_{t \in T_1 \cup T_2})$, $\forall i \in \{1,2\}$. Given $t \in T_2$, let $\gamma_t : (\widehat{M} \times \widehat{\mathcal{F}}^2_{-t}) \times \tilde{A}_1 \twoheadrightarrow \widehat{K}_t$ be the correspondence defined by

$$\gamma_t((m, a_{-t}), z) = \underset{x \in \Psi((m,a_{-t}),z)}{\operatorname{argmax}} \; v_t(x, (m, a_{-t}), z),$$

where[33]

$$\Psi((m, a_{-t}), z) = \lambda(z)\Gamma_t^1(m, a_{-t}) + (1 - \lambda(z))\Gamma_t^2(m, a_{-t}),$$
$$v_t(x, (m, a_{-t}), z) = \lambda(z)u_t^1(m, x, a_{-t}) + (1 - \lambda(z))u_t^2(m, x, a_{-t}).$$

It follows that γ_t is upper hemicountinuous with non-empty and compact values. Therefore, the correspondence $\gamma : (\widehat{M} \times \widehat{\mathcal{F}}^2) \times \tilde{A}_1 \twoheadrightarrow \Pi_{t \in T_2} \widehat{K}_t$ given by

[32] These properties are the core of the proof of equilibrium existence of Riascos and Torres-Martínez (2013).

[33] Remember that $\lambda(z) = \frac{d(z,\overline{U}_2)}{d(z,\overline{U}_1)+d(z,\overline{U}_2)}$.

$\gamma((m,a),z_2) = \prod_{t\in T_2}\gamma_t((m,a_{-t}),z)$ is upper hemicontinuous with compact and non-empty values. In particular, γ has closed graph.

Since $\{(z_1^n,z_2^n,a^n)\}_{n\in\mathbb{N}} \subset \text{Graph}(\gamma)$, it follows that $\tilde{a} \in \gamma(\tilde{z}_1,\tilde{z}_2)$.

On the other hand, for each $n \in \mathbb{N}$, there exists $f_n : T_1 \to \widehat{K}$ such that, $m_n = \int_{T_1} H(t, f_n(t))d\mu$ and, for almost all $t \in T_1$, $f_n(t) \in \xi_t(z_1^n, z_2^n) := \text{argmax}_{x\in\Psi(z_1^n,z_2^n)} v_t(x, z_1^n, z_2^n)$, where we use notations analogous to those described above. Note that, for all $t \in T_1$, ξ_t, it has a closed graph.

Since $m^n \to \tilde{m}$, analogous arguments to those made in Theorem 1 (Claim A) ensure that, applying the multidimensional Fatou's Lemma [see Hildenbrand (1974, page 69)], there exists a full measure set $\dot{T}_1 \subseteq T_1$ and a function $\overline{f} : T_1 \to \widehat{K}$ such that,

(i) For any $t \in \dot{T}_1$, there is a subsequence of $\{f_n(t)\}_{n\in\mathbb{N}}$ that converges to $\overline{f}(t)$;
(ii) For any $t \in T_1\backslash\dot{T}_1$, $\overline{f}(t) \in \xi_t(\tilde{z}_1,\tilde{z}_2)$;
(iii) $\tilde{m} = \int_{T_1} H(t,\overline{f}(t))d\mu$.

Since for any $t \in \dot{T}_1$ the correspondence ξ_t is closed, it follows from item (i) that $\overline{f}(t) \in \xi_t(\tilde{z}_1,\tilde{z}_2)$. Items (ii) and (iii) jointly with the fact that $\tilde{a} \in \gamma(\tilde{z}_1,\tilde{z}_2)$ imply that $(\tilde{m},\tilde{a}) \in \Theta_1(\tilde{z}_1,\tilde{z}_2)$. □

Since $(\overline{m},\overline{a}) \in U_1$, it follows that $G(\overline{m},\overline{a}) = \mathcal{G}_1$. Hence, Claim B implies that $\Lambda(G(\overline{m},\overline{a})) \cap U_1 = \emptyset$. This is a contradiction, since both $(\overline{m},\overline{a}) \in U_1$ and $(\overline{m},\overline{a}) \in \Lambda(G(\overline{m},\overline{a}))$. Therefore, the minimal essential set $m(\mathcal{G})$ is connected.

(iv) If $\Lambda(\mathcal{G})$ is finite, then $\mathcal{G}$ has at least one essential equilibrium.

Suppose that $\widehat{K}$ is a convex subset of a normed space with a metric induced by a norm. It follows from (iii) that for every $\mathcal{G} \in \mathbb{G}'$ there is a minimal essential set of $\Lambda(\mathcal{G})$ that is connected. On the other hand, as $\Lambda(\mathcal{G})$ is finite, minimal essential sets are singletons. □

Proof of Theorem 4 Given $\mathcal{X} \in \mathbb{X}$, the $\mathcal{T}$-essential subsets of $\Lambda(\kappa(\mathcal{X}))$ are stable.

It follows from Definition 4 that it suffices to guarantee that minimal essential sets are stable in the sense of Definition 8. Let $\Lambda_m(\mathcal{T},\mathcal{X})$ be the collection of minimal $\mathcal{T}$-essential subsets of $\Lambda(\kappa(\mathcal{X}))$.

By contradiction, assume that there is $A \in \Lambda_m(\mathcal{T},\mathcal{X})$ and $\epsilon_0 > 0$ such that, for any $\delta > 0$, there is $\mathcal{X}_\delta \in \mathbb{X}$ with $\tau(\mathcal{X},\mathcal{X}_\delta) < \delta$ and $A' \cap B[\epsilon_0, A]^c \neq \emptyset$, $\forall A' \in \Lambda_m(\mathcal{T},\mathcal{X}_\delta)$, where $B[\epsilon_0,A]^c := (\widehat{M}\times\widehat{\mathcal{F}}^2)\backslash B[\epsilon_0,A]$. Since A is $\mathcal{T}$-essential, there is $\delta_0 > 0$ such that, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X},\mathcal{X}') < \delta_0$, we have that $\Lambda(\kappa(\mathcal{X}')) \cap C(\epsilon_0, A) \neq \emptyset$, where $C(\epsilon_0,A) = \{(m,a) \in \widehat{M}\times\widehat{\mathcal{F}}^2 : \inf_{(m',a')\in A} \widehat{\sigma}((m,a),(m',a')) < \epsilon\}$. It follows that $\Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0,A]$ is a non-empty and closed set contained in $B[\epsilon_0,A]$. Therefore, $\Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0,A]$ is not an essential subset of $\Lambda(\kappa(\mathcal{X}_{\delta_0}))$.

Hence, there exists $\epsilon_1 > 0$ such that, for any $n \in \mathbb{N}$, there is $\mathcal{X}_n \in \mathbb{X}$ with $\tau(\mathcal{X}_{\delta_0},\mathcal{X}_n) < \frac{\delta_1}{n}$ and $C(\epsilon_1, \Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0,A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, where $\delta_1 > 0$ is chosen in such form that, for any $\mathcal{X}'' \in \mathbb{X}$, if $\tau(\mathcal{X}_{\delta_0},\mathcal{X}'') < \delta_1$, then $\tau(\mathcal{X},\mathcal{X}'') < \delta_0$. The last property ensures that $\tau(\mathcal{X},\mathcal{X}_n) < \delta_0$ for any $n \in \mathbb{N}$, which implies that $\Lambda(\kappa(\mathcal{X}_n)) \cap C(\epsilon_0,A)$ is non-empty. Take a sequence $\{(m_n,a_n)\}_{n\in\mathbb{N}}$ such that $(m_n,a_n) \in \Lambda(\kappa(\mathcal{X}_n)) \cap C(\epsilon_0,A)$, $\forall n \in \mathbb{N}$. Without the loss of generality, there is $(m_0,a_0) \in B[\epsilon_0,A]$ such that $(m_n,a_n) \to_n (m_0,a_0)$.

The upper hemicontinuity of $(\Lambda \circ \kappa)$ ensures that $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}_{\delta_0}))$, that is, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0, A]$. However, as for any $n \in \mathbb{N}$, we have that $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}_n))$ and $C(\epsilon_1, \Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0, A]) \cap \Lambda(\kappa(\mathcal{X}_n)) = \emptyset$, it follows that $(m_n, a_n) \notin C(\epsilon_1, \Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0, A])$, $\forall n \in \mathbb{N}$. Thus, $(m_0, a_0) \notin \Lambda(\kappa(\mathcal{X}_{\delta_0})) \cap B[\epsilon_0, A]$, which is a contradiction.

If $\mathbb{X}$ is a convex subset of a normed space and τ is induced by a norm, then for each $\mathcal{X} \in \mathbb{X}$, the $\mathcal{T}$-essential components of $\Lambda(\kappa(\mathcal{X}))$ are strongly stable.

Since $\widehat{M} \subset \mathbb{R}^m$ is compact and $\widehat{K}_t$, with $t \in T_2$, are compact subsets of normed spaces with metrics induced by norms, it follows that $\Lambda(\kappa(\mathcal{X})) \subseteq \widehat{M} \times \widehat{\mathcal{F}}^2$ is a locally connected and compact space. Therefore, $\Lambda(\kappa(\mathcal{X})$ as a finite number of connected components.[34] For this reason, given a $\mathcal{T}$-essential component $A \subseteq \Lambda(\kappa(\mathcal{X}))$, there exists $\pi > 0$ such that $B[\pi, A] \cap B[\pi, \Lambda(\kappa(\mathcal{X}))\backslash A] = \emptyset$.

Furthermore, it follows from the proof of Theorem 2(i) that there is $A_m \in \Lambda_m(\mathcal{T}, \mathcal{X})$ such that $A_m \subseteq A$. By the previous item, for each $\epsilon > 0$, there is $\delta_1 > 0$ such that, given $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$, there exists $A'_m \in \Lambda_m(\mathcal{T}, \mathcal{X}')$ for which $A'_m \subseteq B[\epsilon, A_m] \subseteq B[\epsilon, A]$. Since $\mathbb{X}$ is a convex subset of a normed space and τ is induced by a norm, it follows from Theorem 3(iv) that minimal essential sets are connected, and therefore, following analogous arguments to those made in the proof of Theorem 2(ii), we can ensure that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_1$, there is an essential component $A' \in \Lambda_c(\mathcal{T}, \mathcal{X}')$ which contains A'_m, where $\Lambda_c(\mathcal{T}, \mathcal{X}')$ is the set of $\mathcal{T}$-essential components of $\Lambda(\kappa(\mathcal{X}'))$. We want to prove that, for $\mathcal{X}'$ closely enough to $\mathcal{X}$, $A' \subseteq B[\epsilon, A]$.

Since the correspondence $\Lambda \circ \kappa$ is upper hemicontinuous, there is $\delta_2 > 0$ such that for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta_2$ we have that $\Lambda(\kappa(\mathcal{X}')) \subset C(\epsilon, \Lambda(\kappa(\mathcal{X}))) \subset B[\epsilon, A] \cup B[\epsilon, \Lambda(\kappa(\mathcal{X}))\backslash A]$.

Note that $\Lambda(\kappa(\mathcal{X}))\backslash A$ is a compact set.[35] Let $\delta = \min\{\delta_0, \delta_1\}$ and fix $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$. If $A' \cap B[\epsilon, A]^c \neq \emptyset$, then $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X}))\backslash A] \neq \emptyset$ and $A' \cap B[\epsilon, A] \neq \emptyset$. In addition, when $\epsilon < \pi$, it follows that $B[\epsilon, A] \bigcap B[\epsilon, \Lambda(\kappa(\mathcal{X}))\backslash A] = \emptyset$. Since A and $\Lambda(\kappa(\mathcal{X}))\backslash A$ are compact sets, both $B[\epsilon, A]$ and $B[\epsilon, \Lambda(\kappa(\mathcal{X}))\backslash A]$ are closed. Thus, we obtain a partition of the connected set A' into two non-empty and disjoint closed sets, $A' \cap B[\epsilon, \Lambda(\kappa(\mathcal{X}))\backslash A]$ and $A' \cap B[\epsilon, A]$, which is a contradiction. Therefore, for any $\mathcal{X}' \in \mathbb{X}$ with $\tau(\mathcal{X}, \mathcal{X}') < \delta$, we have that $A' \subset B[\epsilon, A]$. □

Lemma 2 *Let $\mathcal{G} = \mathcal{G}((K_t, \Gamma_t, u_t)_{t \in T_1 \cup T_2})$ be a generalized payoff secure and upper semicontinuous game. Then, $\mathcal{G}$ satisfies continuous security.*

Proof Given $(m, a) \notin \Lambda(\mathcal{G})$, generalized payoff security guarantees that, for any $\epsilon > 0$, there exists $(U^\epsilon, (\varphi_t^\epsilon)_{t \in T_1 \cup T_2}, \alpha^\epsilon)$ satisfying item (i) of Definition 10. Thus, to guarantee that $\mathcal{G}$ is continuous secure, it is sufficient to prove that $(U^\epsilon, \alpha^\epsilon)$ satisfies

34 The connected components of a locally connected and compact space determine a partition of it into disjoint open sets. By compactness, this partition has finitely many elements [see Berge (1997, pages 98–100)].

35 Indeed, since $(\Lambda(\kappa(\mathcal{X}))\backslash A) \subset \Lambda(\kappa(\mathcal{X}))$, it is sufficient to ensure that it is closed. Let $\{(m_n, a_n)\}_{n \in \mathbb{N}} \subset (\Lambda(\kappa(\mathcal{X}))\backslash A)$ be a sequence that converges to $(m_0, a_0) \in \widehat{M} \times \widehat{\mathcal{F}}_2$. For any $n \in \mathbb{N}$, $(m_n, a_n) \in \Lambda(\kappa(\mathcal{X}))$ and $(m_n, a_n) \notin A$. Thus, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}))$. Furthermore, if $(m_0, a_0) \in A$, then for n large enough $(m_n, a_n) \in B[\pi, A]$, which is a contradiction with $B[\pi, \Lambda(\kappa(\mathcal{X}))\backslash A] \cap B[\pi, A] = \emptyset$. Therefore, $(m_0, a_0) \in \Lambda(\kappa(\mathcal{X}))\backslash A$.

Definition 10 (ii) for some $\epsilon > 0$. Suppose, by contradiction, that for any $n \in \mathbb{N}$, there is $(f_n, a_n) \in \widehat{\mathcal{F}}^1 \times \widehat{\mathcal{F}}^2$ satisfying,

(a) $(m(f_n), a_n) \in U^{\frac{1}{n}}$,
(b) $f_n(t) \in \Gamma_t(m(f_n), a_n)$ for almost all $t \in T_1$,
(c) $a_{n,t} \in \Gamma_t(m(f_n), a_{n,-t})$ for all $t \in T_2$,
(d) for almost all $t \in T_1$, $u_t(f_n(t), m(f_n), a_n) \geq \alpha^{\frac{1}{n}}(t)$,
(e) for any $t \in T_2$, $u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \alpha^{\frac{1}{n}}(t)$.

Since we can assume that $\bigcap_n U^{\frac{1}{n}} = \{(m, a)\}$, it follows from (a) that $(m(f_n), a_n) \to_n (m, a)$. Conditions (b)-(c) guarantee, by using analogous arguments to those made in the proof of Theorem 1 (Claim A), that we can find a strategy profile $f \in \mathcal{F}^1((K_t)_{t \in T_1})$ such that $m = m(f)$ and there is a full measure set $T_1' \subseteq T_1$ such that $f(t) \in Ls(f_n(t))$, $\forall t \in T_1'$.

In addition, as correspondences of admissible strategies have closed graph, it follows that (i) for almost all $t \in T_1$, $f(t) \in \Gamma_t(m(f), a)$; (ii) for all $k \in T_2$, $a_k \in \Gamma_k(m(f), a_{-k})$.

Hence, as $(m, a) \notin \Lambda(\mathcal{G})$, there is a non-negligible set of agents that are sub-optimizing, i.e., there exists $\delta > 0$ such that either for a positive measure set $T_1'' \subseteq T_1$,

$$u_t(f(t), m, a) + \delta < \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a), \quad \forall t \in T_1'',$$

or for some $t \in T_2$,

$$u_t(m, a_t, a_{-t}) + \delta < \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}).$$

This last condition implies that

$$\sum_{t \in T_2} u_t(m, a_t, a_{-t}) + \delta < \sum_{t \in T_2} \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}).$$

Since $\mathcal{G}$ is upper semicontinuous and $(m(f_n), a_n) \to_n (m, a)$, it follows from the definition of f that for $n \in \mathbb{N}$ large enough, we have that either for all $t \in T_1' \cap T_1''$,

$$u_t(f_n(t), m(f_n), a_n) + 0.5\,\delta < \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a), \tag{1}$$

or

$$\sum_{t \in T_2} u_t(m(f_n), a_{n,t}, a_{n,-t}) + 0.5\,\delta < \sum_{t \in T_2} \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}).$$

The later inequality implies that there is $t \in T_2$ such that

$$u_t(m(f_n), a_{n,t}, a_{n,-t}) + \frac{0.5\,\delta}{\#T_2} < \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}). \tag{2}$$

On the other hand, it follows from conditions (d)–(e) above and Definition 11(ii) that for $n \in \mathbb{N}$ large enough, there exists $T_n \subseteq T_1$ with $\mu(T_n) \geq \mu(T_1) - \frac{1}{n}$ such that, for any $t \in T_n$,

$$u_t(f_n(t), m(f_n), a_n) \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a) - \frac{1}{n},$$

and for every atomic player $t \in T_2$,

$$u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t}) - \frac{1}{n}.$$

Thus, $\underline{\lim}_n u_t(f_n(t), m(f_n), a_n) \geq \sup_{x \in \Gamma_t(m,a)} u_t(x, m, a)$ for almost all non-atomic player $t \in T_1$. Also, for each atomic player $t \in T_2$, we have that $\underline{\lim}_n u_t(m(f_n), a_{n,t}, a_{n,-t}) \geq \sup_{x \in \Gamma_t(m,a_{-t})} u_t(m, x, a_{-t})$. Hence, taking the lower limit in (1) and (2), we obtain a contradiction. □

Proposition 2 *$(\mathbb{G}_d, \rho)$ is a complete metric space.*

Proof Since $(K_t, \Gamma_t)_{t \in T_1 \cup T_2}$ does not change, $(\mathbb{G}_d, \rho)$ can be considered as a subset of the space of bounded functions $\mathcal{B} := \mathcal{U}(T_1 \times \widehat{K} \times \widehat{M} \times \widehat{\mathcal{F}}^2) \times \prod_{t \in T_2} \mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$, where for any $t \in T_2$, the set $\mathcal{U}_t(\widehat{M} \times \widehat{\mathcal{F}}^2)$ is the collection of bounded functions $(m, a_t, a_{-t}) \to u_t(m, a_t, a_{-t})$ which are quasi-concave on a_t. Note that $(\mathcal{B}, \rho)$ is a complete metric space and, therefore, it is sufficient to ensure that $\mathbb{G}_d$ is a closed subset of $\mathcal{B}$.

Fix a sequence $\{\mathcal{G}_n\}_{n \in \mathbb{N}} \subset \mathbb{G}_d$, with $\mathcal{G}_n = \mathcal{G}_n((u_t^n)_{t \in T_1 \cup T_2})$ for any $n \in \mathbb{N}$, which converges to $\overline{\mathcal{G}} = \overline{\mathcal{G}}((\overline{u}_t)_{t \in T_1 \cup T_2}) \in \mathcal{B}$. We want to prove that $\overline{\mathcal{G}} \in \mathbb{G}_d$.
Claim. $\overline{\mathcal{G}}$ is generalized payoff secure.
Given $(m, a) \in \widehat{M} \times \widehat{\mathcal{F}}^2$ and $\epsilon > 0$, generalized payoff security of $\mathcal{G}_n$ at $((m, a), 0.5\,\epsilon)$ implies that there exists $(U^n, (\varphi_t^n)_{t \in T_1 \cup T_2}, \alpha^n)$ satisfying the requirements of Definition 11.

Thus, for n large enough, for almost all $t \in T_1$, for all $k \in T_2$, and for every $(m', a') \in U^n$, we have that,

$$\overline{u}_t(x, m', a') > u_t^n(x, m', a') - 0.25\,\epsilon \geq \alpha^n(t) - 0.25\,\epsilon, \quad \forall x \in \varphi_t^n(m', a');$$
$$\overline{u}_k(m', x, a'_{-t}) > u_k^n(m'.x, a'_{-t}) - 0.25\,\epsilon \geq \alpha^n(k) - 0.25\,\epsilon, \quad \forall x \in \varphi_k^n(m', a').$$

Furthermore, Definition 11(ii) ensures that, for n large enough,

$$\begin{aligned}\big(\alpha^n(k) - 0.25\,\epsilon\big) + \epsilon &= (\alpha^n(k) + 0.5\epsilon) + 0,25\,\epsilon \\ &\geq \sup_{x \in \Gamma_t(m,a_{-k})} u_t^n(m, x, a_{-k}) + 0.25\,\epsilon \\ &> \sup_{x \in \Gamma_t(m,a_{-k})} \overline{u}_t(m, x, a_{-k}), \quad \forall k \in T_2.\end{aligned}$$

$$\mu\left\{t \in T_1 : \big(\alpha^n(t) - 0.25\,\epsilon\big) + \epsilon \geq \sup_{x \in \Gamma_t(m,a)} \overline{u}_t(x, m, a)\right\} \geq \mu(T_1) - 0.5\,\epsilon.$$

Therefore, taking n large enough and choosing $(U^n, (\varphi_t^n)_{t\in T_1\cup T_2}, \alpha^n - 0.25\,\epsilon)$, we ensure that $\overline{\mathcal{G}}$ is generalized payoff secure at $((m,a),\epsilon)$. □

It is a direct consequence of Carbonell-Nicolau (2010, Lemma 1, page 425) that $\overline{\mathcal{G}}$ is upper semicontinuous and atomic players' objective functions $(\overline{u}_t)_{t\in T_2}$ are quasi-concave. In addition, the same arguments made in the proof of Proposition 1 guarantee that, for every $(m,a)\in \widehat{M}\times\widehat{\mathcal{F}}^2$, the map $(t,x)\in T_1\times\widehat{K}\to \overline{u}_t(x,m,a)$ is measurable. This concludes the proof. □

Proof of Theorem 5 Since $(\mathbb{G}_d,\rho)$ is complete, it follows from the proofs of Theorems 1–4 that it is sufficient to ensure that Λ still has a closed graph when its domain is extended to $\mathbb{G}_d$.

Let $\{(\mathcal{G}_n,(m_n,a_n))\}_{n\in\mathbb{N}}\subset \text{Graph}(\Lambda)$ such that $(\mathcal{G}_n,(m_n,a_n))\to(\overline{\mathcal{G}},(\overline{m},\overline{a}))$, where $\mathcal{G}_n=\mathcal{G}_n((u_t^n)_{t\in T_1\cup T_2})$ and $\overline{\mathcal{G}}=\overline{\mathcal{G}}((\overline{u}_t)_{t\in T_1\cup T_2})\in\mathbb{G}_d$. We want to prove that $(\overline{m},\overline{a})\in\Lambda(\overline{\mathcal{G}})$.

Since $(m_n,a_n)\in\Lambda(\mathcal{G}_n)$, there is $f_n\in\widehat{\mathcal{F}}^1$ such that

(a) the function $g_n: T_1\to\mathbb{R}^m$ given by $g_n(t)=H(t,f_n(t))$ is measurable and $m_n=m(f_n)$;
(b) for almost all $t\in T_1$ both $f_n(t)\in\Gamma_t(m_n,a_n)$ and

$$u_t^n(f_n(t),m_n,a_n)=\sup_{x\in\Gamma_t(m_n,a_n)} u_t^n(x,m_n,a_n).$$

Claim A *There exists $\overline{f}\in\widehat{\mathcal{F}}^1$ such that $\overline{m}=\int_{T_1}H(t,\overline{f}(t))d\mu$.*

Proof By analogous arguments to those in the proof of Theorem 1 (Claim A), we can find a strategy profile $\overline{f}\in\widehat{\mathcal{F}}^1$ and a full measure set $T_1^*\subseteq T_1$ such that $\overline{m}=m(\overline{f}),\ \overline{f}(t)\in Ls(f_n(t)),\ \forall t\in T_1^*$. □

Claim B *For almost all $t\in T_1$, $\overline{f}(t)\in\Gamma_t(\overline{m},\overline{a})$. In addition, for any $t\in T_2$, $\overline{a}_t\in\Gamma_t(\overline{m},\overline{a}_{-t})$.*

Proof It follows from the proof of Claim A that there is a full measure set $T_1^*\subseteq T_1$ such that $\overline{f}(t)\in Ls(f_n(t)),\ \forall t\in T_1^*$. Thus, the closed graph property of correspondences of admissible strategies ensures that (i) for all $t\in T_1^*$, $\overline{f}(t)\in\Gamma_t(\overline{m},\overline{a})$; and (ii) for all $t\in T_2$, $\overline{a}_t\in\Gamma_t(\overline{m},\overline{a}_{-t})$. □

Claim C *The following properties hold*

(i) *For almost all $t\in T_1$, $\overline{f}(t)\in argmax_{x\in\Gamma_t(\overline{m},\overline{a})}\ \overline{u}_t(x,\overline{m},\overline{a})$.*
(ii) *For any $t\in T_2$, $\overline{a}_t\in argmax_{x\in\Gamma_t(\overline{m},\overline{a}_{-t})}\ \overline{u}_t(\overline{m},x,\overline{a}_{-t})$.*

Proof Suppose that at least one of the properties (i) and (ii) does not hold, i.e., there is a non-negligible set of agents that are suboptimizing. Since $\overline{\mathcal{G}}$ is upper semicontinuous, identical arguments to those made in the proof of Lemma 2 to obtain conditions (1) and (2) imply that there is $\xi>0$ such that,[36] for n large enough, either

$$\overline{u}_t(f_n(t),m_n,a_n)+\xi<\sup_{x\in\Gamma_t(\overline{m},\overline{a})}\overline{u}_t(x,\overline{m},\overline{a}),\quad \forall t\in T_1^{**}\subseteq T_1,$$

where T_1^{**} is a positive measure set, or there exists $t \in T_2$ such that[36]

$$\overline{u}_t(m_n, a_{n,t}, a_{n,-t}) + \xi < \sup_{x \in \Gamma_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t}).$$

Thus, as $\rho(\mathcal{G}_n, \overline{\mathcal{G}}) \to_n 0$, for n large enough at least one of the following conditions hold:

$$u_t^n(f_n(t), m_n, a_n) + 0.5\,\xi < \sup_{x \in \Gamma_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a}), \quad \forall t \in T_1^* \cap T_1^{**}; \tag{3}$$

$$u_t^n(m_n, a_{n,t}, a_{n,-t}) + 0.5\,\xi < \sup_{x \in \Gamma_t(\overline{m}, \overline{a}_{-t})} \overline{u}_t(\overline{m}, x, \overline{a}_{-t}). \tag{4}$$

On the other hand, since $\overline{\mathcal{G}}$ is a generalized payoff secure large game, for every $\epsilon > 0$, there exists $(U^\epsilon, (\varphi_t^\epsilon)_{t \in T_1 \cup T_2}, \alpha^\epsilon)$ satisfying Definition 11. In particular, as $(m_n, a_n) \to_n (\overline{m}, \overline{a})$, there exists a set $T_\epsilon \subseteq T_1$ with $\mu(T_\epsilon) \geq \mu(T_1) - \epsilon$ such that, for any n large enough, we have $(m_n, a_n) \in U^\epsilon$ and the following properties hold for every $(t, k) \in T_\epsilon \times T_2$:

$$\begin{aligned} \sup_{x \in \Gamma_t(m_n, a_n)} \overline{u}_t(x, m_n, a_n) &\geq \sup_{x \in \varphi_t^\epsilon(m_n, a_n)} \overline{u}_t(x, m_n, a_n) \\ &\geq \alpha^\epsilon(t) \geq \sup_{x \in \Gamma_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a}) - \epsilon; \end{aligned} \tag{5}$$

$$\begin{aligned} \sup_{x \in \Gamma_k(m_n, a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}) &\geq \sup_{x \in \varphi_k^\epsilon(m_n, a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}) \\ &\geq \alpha^\epsilon(k) \geq \sup_{x \in \Gamma_k(\overline{m}, \overline{a}_{-k})} \overline{u}_k(\overline{m}, x, \overline{a}_{-k}) - \epsilon. \end{aligned} \tag{6}$$

As objective functions are bounded, the uniform convergence of $\mathcal{G}_n$ to $\overline{\mathcal{G}}$ ensures that, for n large enough and for each $(t, k) \in T_1^* \times T_2$,

$$\begin{aligned} u_t^n(f_n(t), m_n, a_n) + \epsilon &= \sup_{x \in \Gamma_t(m_n, a_n)} u_t^n(x, m_n, a_n) + \epsilon \\ &\geq \sup_{x \in \Gamma_t(m_n, a_n)} \overline{u}_t(x, m_n, a_n); \end{aligned} \tag{7}$$

$$\begin{aligned} u_k^n(m_n, a_{n,k}, a_{n,-k}) + \epsilon &= \sup_{x \in \Gamma_k(m_n, a_{n,-k})} u_k^n(m_n, x, a_{n,-k}) + \epsilon \\ &\geq \sup_{x \in \Gamma_k(m_n, a_{n,-k})} \overline{u}_k(m_n, x, a_{n,-k}). \end{aligned} \tag{8}$$

[36] Following the notation used in the proof of Lemma 2, we can take $\xi = \frac{0.5\,\delta}{\# T_2}$.

Therefore, for every $\epsilon > 0$ and for each $(t, k) \in (T_1^* \cap T_\epsilon) \times T_2$, taking the lower limit as n goes to infinity on inequalities (7)–(8) it follows that,

$$\underline{\lim}_n\, u_t^n(f_n(t), m_n, a_n) + \epsilon \geq \sup_{x \in \Gamma_t(\overline{m}, \overline{a})} \overline{u}_t(x, \overline{m}, \overline{a}) - \epsilon, \tag{9}$$

$$\underline{\lim}_n\, u_k^n(m_n, a_{n,k}, a_{n,-k}) + \epsilon \geq \sup_{x \in \Gamma_k(\overline{m}, \overline{a}_{-k})} \overline{u}_k(\overline{m}, x, \overline{a}_{-k}) - \epsilon. \tag{10}$$

Taking the limit as ϵ goes to zero on (9–10) and taking the lower limit as n goes to infinity on (3–4), we obtain a contradiction.

It follows from Claims A and C that $(\overline{m}, \overline{a}) \in \Lambda(\overline{\mathcal{G}})$. □

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